



GAIMME

**GUIDELINES FOR ASSESSMENT & INSTRUCTION
IN MATHEMATICAL MODELING EDUCATION**

**CONSORTIUM FOR MATHEMATICS AND
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PREFACE

INTRODUCTION

In 2015, leaders from SIAM and COMAP came together to produce a report we have named GAIMME – Guidelines for Assessment and Instruction in Mathematical Modeling Education. The name gives homage to the fine work of the American Statistical Association’s impressive GAISE report.¹ Like that document our primary audience is you the teacher. While we hope that test and policy makers will read this document and use it in their decision-making, it has been written for the front-line teacher. We hope and intend that these guidelines will be of help as you incorporate the practice of mathematical modeling into your classrooms.

A major reason for the creation of GAIMME was the fact that, despite the usefulness and value in demonstrating how mathematics can help analyze and guide decision making for real world messy problems, many people have limited experience with math modeling. We wanted to paint a clearer picture of mathematical modeling (what it is and what it isn’t) as a process and how the teaching of that process can mature as students move through the grade bands, independent of the mathematical knowledge they may bring to bear.

HOW TO USE THIS DOCUMENT

While we very much hope that you have the time and interest to read this document cover to cover, it has been written to permit a more focused reading as well. The opening section, “What is Mathematical Modeling?” is intended for every reader and contains the main arguments for the inclusion of mathematical modeling across all grade levels. It is a necessary introduction to each of the succeeding sections. This chapter is followed by three grade-level chapters: early grades, high school, and undergraduate. As a teacher you may wish, naturally, to proceed from “What is Mathematical Modeling?” to the relevant grade level section. However, we encourage you to look at preceding and/or subsequent sections to get a clearer idea of how the modeling process can progress from grade level to grade level.

While these grade level sections have naturally a somewhat different voice, they have important common elements. Each begins by making the case for teaching mathematical modeling at that level. Where relevant, we discuss how teaching the mathematical modeling process at a particular level differs from how that process was treated at the prior level. This is independent of the mathematical topics brought to bear in creating the models.

Each section contains a collection of exemplary tasks. These tasks have been chosen for one or more of the following reasons:

- They are interesting and/or important for students to experience.
- They exemplify specific components of the modeling cycle.
- They are doable by real students in real classrooms in real time.

Because modeling problems can be extremely rich and most often require reexamining our assumptions, they can require time (and pages) to explore. For this reason we have included several of the longer tasks and explanations in Appendices B and C.

In each grade level section we spend time discussing both the teaching of mathematics through the modeling process and the teaching of the modeling process itself. In addition we discuss the characteristics of appropriate assessments. Because of this structure you may find some duplication of ideas. This is intentional. While there are certain differences in how one teaches mathematical modeling from grade to grade, there are a number of important features they share in common and that deserve emphasis.

Following the educational level sections we revisit the question of “What is Mathematical Modeling” through a series of quotes and comments from the author team. This section, intended for all readers, is included to emphasize the art and flavor of modeling from the perspective of practitioners and mathematics educators.

The report includes a Resources Appendix A, designed to sample some of the important modeling assessments and curricula materials in wide usage. A discussion of what makes a good modeling assessment is also included in the Resources section, along with a discussion of how to adapt assessments to meet different modeling goals. We conclude with an Appendix D, which contains helpful examples of rubrics and assessment tools which may be used when teaching mathematical modeling or teaching mathematics through modeling.

A few things that the GAIMME report is not: It is not intended as a curriculum. It does not contain a full set of modeling problems. It is not designed as a collection of lesson plans for immediate classroom use, nor a complete set of classroom assessments. Rather, the purpose of these guidelines is to give a sense what mathematical modeling is and isn't and practical advice on how to teach modeling through the grade levels. We have tried to demonstrate the importance of mathematical modeling and how and why it should be an essential part of every student's mathematics experience throughout their education.



WHAT IS MATHEMATICAL MODELING?

INTRODUCTION

In the course of a student’s mathematics education, the word “model” is used in many ways. Several of these, such as manipulatives, demonstration, role modeling, and conceptual models of mathematics, are valuable tools for teaching and learning. However, they are different from the practice of mathematical modeling. Mathematical modeling, both in the workplace and in school, uses mathematics to answer big, messy, reality-based questions. Speaking in this report to our colleagues as teachers, supervisors, and teacher educators, we share our experience in the classroom so that, with appropriate facilitation from teachers, students can engage in genuine modeling activities and use mathematics to answer questions that they find meaningful and will enhance their futures.

The authors of this report firmly believe that mathematical modeling should be taught at every stage of a student’s mathematical education. After all, why does society give us so much time to teach mathematics? In part, it is because mathematics is important for its own sake, but mostly because mathematics is important in dealing with the rest of the world. Certainly mathematics will help students as they move on through school and into the world of work. But it can and should help them in their daily lives and as informed citizens. It is crucial that students’ experiences with mathematical modeling, as they progress through the grades, give them exposure to a wide variety of problems — how do we determine the average rainfall in a state? Where’s the best place to locate a fire station? What is a fair voting system? How can I hang pictures along a staircase so they look straight? As we demonstrate in subsequent sections of this report, students can learn and appreciate the importance of modeling in their lives at all educational levels.

**BUT WHAT IS
MATHEMATICAL
MODELING?**

In the sections below we will provide several examples of what modeling is and isn't. We will give a more detailed working definition of mathematical modeling. But to describe it in a nutshell:

Mathematical modeling is a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena.

Most short definitions we find emphasize this most important aspect, namely the relation between modeling and the world around us.

- Using the language of mathematics to quantify real-world phenomena and analyze behaviors.
- Using math to explore and develop our understanding of real world problems.
- An iterative problem solving process in which mathematics is used to investigate and develop deeper understanding.

Let's consider how we can translate this into our classrooms.

**FROM
MATHEMATICS
SETTINGS TO
APPLICATIONS TO
MODELING
PROBLEMS**

Mathematical modeling can be used to motivate curricular requirements and can highlight the importance and relevance of mathematics in answering important questions. It can also help students gain transferable skills, such as habits of mind that are pervasive across subject matter. We will give a more precise definition of mathematical modeling in the next section; for now we focus, by way of examples, on how elements of mathematical modeling can be incorporated into existing curricula by making small changes to familiar application or story problems.

AT THE PREKINDERGARTEN-GRADE 8 LEVEL

In this example, students are learning the skill of addition. An addition problem, devoid of application, could ask students to 'compute the sum of $6+3$ '. A word problem might add labels and ask the students how many pretzels Jim and Suzy have combined if Jim has six pretzels and Suzy has three.

While the word problem attaches objects to the addition, it is not a modeling problem for two reasons. First, it does not have intrinsic value or meaning for students. Other than answering this problem for homework, why would students care about how many pretzels Jim and Suzy have? The word problem is closed at the beginning and the end. While students may use a few different approaches to reach the answer, such as drawing a picture and counting or writing and evaluating an arithmetic expression, all of the necessary data is clearly provided, and there is only one correct solution.

Let's explore how you could transform this word problem into a mathematical modeling

problem while still engaging students in whole number addition. In order to have students engage with the context more, you could ask the students to imagine that they are helping pack a picnic lunch for their family, and they have to determine how many pretzels they need to pack. Students will need to take lots of things in to consideration and can ask themselves many questions. How many people are in their family, and how many will be attending the picnic? How many pretzels is each person likely to eat? Does anyone have dietary restrictions that should be considered? How much other food will be available at the picnic?

Students will have to make assumptions about the context; their input about their family matters and changes the answer to the question. After they have answered those questions

and made assumptions, however, they still must add whole numbers. The small change of having students think about the context to determine what numbers to add, and even how many numbers to add, transforms the closed question into an open-ended question. The revised question invites students to become a part of the context, while still targeting the same mathematics curriculum content that the students need to learn.

Mathematical modeling can be used to motivate curricular requirements and can highlight the importance and relevance of mathematics in answering important questions.

AT THE HIGH SCHOOL LEVEL

In this example, students are learning how to write equations and graphs given slope and vertical intercept. A problem devoid of application might ask students to graph the line with slope 2 and vertical intercept 100, and then write the equation of the line.

A first move towards modeling might be to contextualize such a problem:

Emily works at a retail store that pays her \$100 per week plus \$2 for each item she sells. Write and graph a linear equation representing the relationship between Emily's weekly income and the number of items she sells during one week.

A student reading this question might ask, "Why would I want to make this graph?" In this instance, the attempt to contextualize the mathematics may actually make the mathematical work seem completely irrelevant.

We can transform this problem into more open-ended and meaningful problem by changing it slightly, as follows:

The holidays are approaching and your best friend Karen would like to make some money to purchase gifts. She found one job that will pay \$2/hr above the minimum

wage. Another job offers to pay half the minimum wage plus commission in the amount of \$2 per item she sells. Which job is better? To help Karen understand your analysis, include a useful representation to help her make the decision.

Students will have to do a little work to answer this question. They might need to look up the minimum wage. They will have to think about the “break-even” point – the number of items their friend will have to sell every hour in order to earn minimum wage. Then they will have to think about whether it is likely Karen would sell that many items, which probably depends on the item and her personality. Research into a context and assumptions about the context are components of mathematical modeling.

Their opinions matter and influence the answer to this question.

However, even that is not enough for students to answer this question. A risk-averse student might advise Karen to take the first job because the pay is decent and is guaranteed. Alternately, a risk-seeking student might advise her to take the second job because of the possibility of making much more money. Determining the break-even point is only one aspect of this question; students will have to think about making decisions in the face of uncertainty. Their opinions matter and influence their answer to this question. They still have to do the same mathematics to answer the question, but they are forced to reconcile their answer with reality, making the mathematics more relevant and interesting. Making judgments about what matters and assessing the quality of a solution are components of mathematical modeling.

AT THE UNDERGRADUATE LEVEL

Early in a first semester calculus course, students often learn how to approximate a derivative by computing the difference quotient, $[f(x+h) - f(x)]/h$, using decreasing step sizes. For example, a textbook problem may assume students know the difference quotient, and look like this:

Approximate the derivative of the function $f(x) = x^3e^x$ at the point $x = 2$ by computing the difference quotient with $h = 0.1$, $h = 0.01$, and $h = 0.001$.

We might transform this type of problem into an application problem by adding context. We might even present information based on real data:

Lapse rates can help identify pockets of unstable air and are particularly important for the flight of unmanned aircraft. The lapse rate, γ , is defined as rate at which the temperature decreases as altitude increases, or $\gamma = -dT/da$, where T is the temperature and a is altitude. The temperature of the atmosphere over Little Rock, Arkansas on a typical October day has been measured and is approximately given by the function:

$T(a) = 24.3 - 5.81a + 0.295a^2 - 0.057a^3 + 0.0024a^5 + 0.006\cos(a)$, where T is in degrees centigrade and a is in kilometers.

Approximate the lapse rate at a height of 11km by computing the difference quotient with $h = 0.1$, $h = 0.01$, and $h = 0.001$.

Although the application problem does offer context, the problem does not give any opportunity for students to put their analysis back into context to answer a larger question. The problem is still a closed problem. A mathematical modeling question should force students to take ownership of some of the decisions along the way. Below is one way in which the above word problem could be modified to turn it into a modeling problem while still having students engage with the idea of using difference quotients to approximate a derivative.

Lapse rates can help identify pockets of unstable air and are particularly important for the flight of unmanned aircraft. The lapse rate, γ , is defined as rate at which the temperature decreases as altitude increases, or $\gamma = -dT/da$, where T is the temperature and a is altitude. The temperature of the atmosphere over Little Rock, Arkansas on a typical October day has been measured and is approximately given by the function: $T(a) = 24.3 - 5.81a + 0.295a^2 - 0.057a^3 + 0.0024a^5 + 0.006\cos(a)$, where T is in degrees centigrade and a is in kilometers.

You have been asked to assess the safety of a potential drone surveillance mission over the city of Little Rock. The drone is supposed to fly at an approximate height of 11km over the city, although the mission could be accomplished at any altitude between 9km and 15km. Your drone can definitely fly safely if the lapse rate is less than $6^\circ\text{C}/\text{km}$; if the lapse rate is over $8^\circ\text{C}/\text{km}$, then your drone must be grounded for safety reasons. Using what you know about difference quotients, assess the safety of the mission and offer a full set of recommendations to your supervisor along with any assumptions you made in completing your analysis.

Unlike the application problem, this modeling version does not tell students which values of h to use; they must choose a reasonable value on their own. Perhaps in exploring values, they will internalize the idea that the approximation improves with decreasing values of h , in preparation of the limit definition of the derivative.

This modeling version of the problem also has students interpret their mathematical results in the context of the drone mission's safety while drawing conclusions and making recommendations based on those results. When (if) they get a mathematical conclusion that falls between the safe and unsafe zones, they will need to make decisions about how to proceed, such as exploring other possible altitudes for the mission. Regardless of what students decide, they need to justify those decisions, requiring them to truly own the mathematical processes that led to the final result.



FIGURE 1.1: ONE WAY OF TRANSFORMING A MATHEMATICS PROBLEM INTO A MODELING PROBLEM.

As demonstrated by the three grade-level specific examples above, a mathematical problem can be transformed into a modeling problem. It is important to see that adding labels, such as “pretzels,” is not sufficient. Perhaps less obvious but equally important, adding context and meaning, such as jobs and wages or lapse rates and drones, is also not sufficient. A modeling problem must also provide room for students to interpret the problem and have choices in the solution process. These ideas for how to transform questions are illustrated in Figure 1.1.

**THE MODELING
PROCESS**

For the purpose of this report we will say that Mathematical Modeling is a process made up of the following components:

IDENTIFY THE PROBLEM

We identify something in the real world we want to know, do, or understand. The result is a question in the real world.

MAKE ASSUMPTIONS AND IDENTIFY VARIABLES

We select ‘objects’ that seem important in the real world question and identify relations between them. We decide what we will keep and what we will ignore about the objects and their interrelations. The result is an idealized version of the original question.

DO THE MATH

We translate the idealized version into mathematical terms and obtain a mathematical formulation of the idealized question. This formulation is the model. We do the math to see what insights and results we get.

ANALYZE AND ASSESS THE SOLUTION

We consider: Does it address the problem? Does it make sense when translated back into the real world? Are the results practical, the answers reasonable, the consequences acceptable?

ITERATE

We iterate the process as needed to refine and extend our model.

IMPLEMENT THE MODEL

For real world, practical applications, we report our results to others and implement the solution.

Mathematical modeling is often pictured as a cycle, since we frequently need to come back to the beginning and make new assumptions to get closer to a useable result. However, we will use the representation below as it reflects the fact that in practice a modeler often bounces back and forth through the various stages:

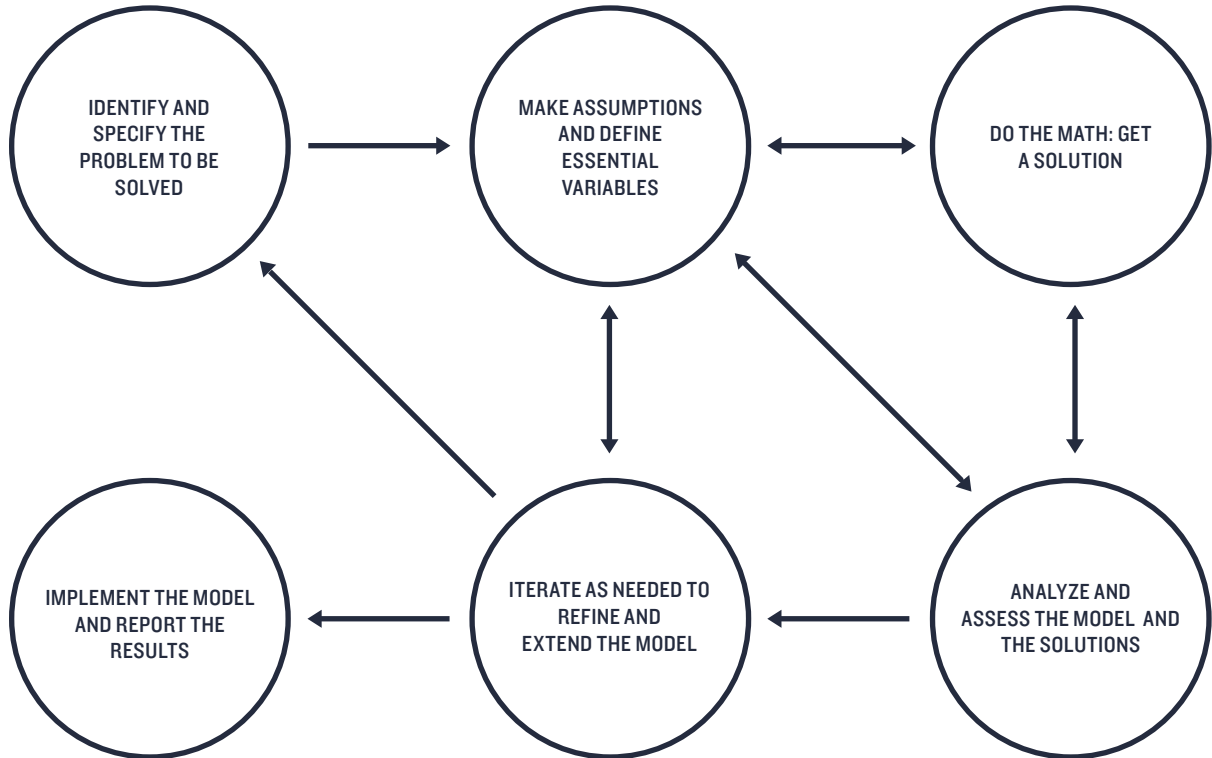


FIGURE 1.2: THE MATH MODELING PROCESS

Note that as shown in the figure, the modeling process contains cyclical components and that not all arrows are unidirectional. As such we have intentionally not used the term ‘steps’ nor have we numbered the components of the modeling process. We do not wish to imply that there exists an ordered number of steps that we could follow to guarantee that we have found a solution to a modeling problem. On the contrary some components happen in parallel and some are repeated as needed. We will see some of this nuance in the examples below and others are highlighted in the examples in the Appendices.

Throughout this report, this will be our working definition. However, we will keep in mind that there is an art to modeling (discussed more deeply in Chapter 5) and to the teaching of modeling, something that we very much hope to capture as we explore implementation through the grade levels.

**THE
MATHEMATICAL
MODELING
PROCESS IN
ACTION**

Perhaps the best way to discuss the components of the modeling process is by way of example. While we may focus only on pieces of the modeling process at particular times in our classrooms, it is important to have an awareness of how these components fit in to the entire modeling process. Understanding of how the process works as a whole can inform the way we support students in engaging with some smaller subset of the process. In the long-term, we hope students are able to use this modeling process to address big, real-world, messy complicated questions whose answers really matter. Helping students build confidence in their ability to complete individual components of the modeling process means that they may be less intimidated to engage with the process as a whole some day.

It is critical to note that *example problems should NOT be used as a template* for all modeling problems. Real-world problems do not all come packaged the same way, so the best thing we can do (for ourselves and our students!) is to focus on the process, and not on the specific content presented here. In the later grade-specific sections, you can find more examples of level-appropriate questions as well as discussions about your role as the teacher throughout the process.

**REAL WORLD
PROBLEM**

Consider the following example with a question many people ask themselves on a daily basis.

Gas prices change on a nearly daily basis, and not every gas station offers the same price for a gallon of gas. The gas station selling the cheapest gas may be across town from where you are driving. Is it worth the drive across town for less expensive gas? Create a mathematical model that can be used to help understand under what conditions it is worth the drive.

Perhaps your initial reaction to this question is that you have not been given enough information to answer the question. This is precisely the point! Most word problems students encounter give them all the information they need to answer the question, but this is not the type of problem that students are likely to encounter when they enter the working world. It is not unrealistic to imagine a student's future boss asking for a solution to achieve a goal in the safest, most timely, and most cost-effective fashion, and for this problem, the employee will need to identify the information they need and seek it out in the process of answering the question. In a similar fashion, the intentional vagueness of the posed modeling question prepares students to think about what information they will need in

order to be able to answer the question.

We will now trace the thought-process of a hypothetical team of students, known as Team A, as they work through this gas problem. Because the purpose of this example is to highlight the modeling process, we showcase particularly good work, so the work we will see from Team A is representative of a strong group of high school students who are all experienced in modeling. They will demonstrate both persistence and self-correction, and they will produce a thoughtful yet imperfect solution. Note that we will not judge the quality of their assumptions and decisions. Instead, the members of Team A will perform their own assessment in the Model Analysis and Assessment component of the modeling process.

**BUILDING THE
MODEL: DEFINING
THE PROBLEM,
MAKING
ASSUMPTIONS,
AND DEFINING
VARIABLES**

Team A needs to distill the given question, “Is it worth it to drive across town to buy gas?” down to a tractable, well-defined question they can answer. Some of their assumptions will come easily, and defining variables follows naturally. The team does not sit down with a picture of the modeling process and consciously work through the components one by one. Instead, they do some research and bounce ideas off one another, and the process of building the model comes organically. They take notes throughout these preliminary conversations, in which they capture the resulting model components they have developed.

In an effort to get a handle on the problem, Team A immediately starts looking up facts. They choose the route that one of them takes to get home which passes by a gas station, which we will call Station 1. A quick internet search yields gas prices at nearby gas stations, so they jot down the price of gas, \$3.59/gallon, at Station 1. Then they look at the other gas stations and choose one, which we will call Station 2, which has the cheapest gasoline price, at \$3.44/gallon.

Next they decide that in order to make a fair comparison, they will need to buy equal amounts of gas. They arbitrarily choose that they will need to purchase 15 gallons of gas. They then write the following:

$$\begin{aligned}\text{Cost of gas at Station 1} &= (\$3.59/\text{gallon}) * 15 \text{ gallons} = \$53.85 \\ \text{Cost of gas at Station 2} &= (\$3.44/\text{gallon}) * 15 \text{ gallons} = \$51.60 \\ \text{Savings by purchasing gas at Station 2} &= \$53.85 - \$51.60 = \$2.25\end{aligned}$$

Therefore, it costs \$2.25 less to purchase gas at the station which charges less.

Then one student realizes that this does not really answer the question, and that they need to consider the added cost of getting to Station 2.

Another student immediately does another internet search and determines that the

distance between the gas stations is 6 miles. They need to figure out how much it will cost to pay for the gas it will take to travel 6 miles to Station 2 and then 6 miles back.

A team member reports having a conversation with his father about a sports car he liked. His dad said it was a great car, but that it did not get good mileage. The father said his car got about 35 miles per gallon, whereas the sports car probably got about 15 miles per gallon. The team realized, by looking at the units involved, that if they divided the 12 miles by the fuel economy, they could get the number of gallons of gas needed to travel the 12 miles. So they write:

$$\begin{aligned} \text{(Cost to drive 12 miles)} &= \frac{12 \text{ miles}}{35 \text{ miles/gallon}} \\ &= 0.34 \text{ gallons} \end{aligned}$$

As soon as they write it down, they realize they are missing something. They correct their work by multiplying by the cost of gas at Station 2 (where, they reason, they will be if they go these extra 12 miles), as follows:

$$\begin{aligned} \text{(Cost to drive 12 miles)} &= \frac{12 \text{ miles}}{35 \text{ miles/gallon}} * \frac{\$3.44}{\text{gallon}} \\ &= 0.34 \text{ gallons} * \frac{\$3.44}{\text{gallon}} \\ &= \$1.18 \end{aligned}$$

Now, they conclude, it is clear that driving the extra 12 miles costs \$1.18, but then they can save \$2.25 when they purchase gas. Therefore, they save \$1.07 by driving across town to Station 2.

At this point, Team A thinks they have answered the question. They have, in fact, done the important work. They have decided which factors matter and they have found ways to relate the things they know to things they want to know.

When they go back and read the question, however, they realize they have only answered the question for one very specific case; they used particular values for the price of gasoline at each station, the particular distance between two particular stations, and the fuel economy for a specific vehicle. However, their previous work is not wasted; they need only generalize the work they have already done. They go back to their work and now use variables to represent the quantities.

At the end of their discussion, Team A has written down the following summary of their work:

DEFINING THE PROBLEM

Determine which costs less

- Purchasing gas at Gas Station 1, which is on our planned route, or
- Traveling out of our way to Gas Station 2 (which sells gas at a cheaper rate) to purchase gas there.

ASSUMPTIONS

- Gas costs less at the gas station that is out of our way.
- The fuel economy of the car remains constant.
- If we choose to go out of our way, we will consider the added cost of the mileage between the gas station we would have gone to and the further station, and back.
- We will purchase the same amount of gas, no matter which gas station we choose.

DEFINING VARIABLES

- Let m be the number miles between Station 1 and Station 2
- P_1 , the price of gasoline at Station 1, in dollars per gallon
- P_2 , the price of gasoline at Station 2, in dollars per gallon
- f , the fuel economy of the car, in miles per gallon
- n , the number of gallons of gas to be purchased
- T , the cost to travel to and from Station 2, in dollars
- S , the difference in money paid for purchasing the gasoline at Station 2 versus Station 1 (accounting only for the purchase of gasoline; not including the travel costs), in dollars

TEAM A'S RESULTING MODEL

- $S = (P_1 - P_2) * n$
- $T = (P_2 * 2m) / f$
- If $S > T$, then it is better to drive across town to purchase gas at Station 2.
- If $S \leq T$, then we should not drive across town to purchase gas.

GETTING A SOLUTION

In some sense, Team A already found a solution when they used particular values earlier. However, that really got them thinking about bigger, deeper questions. For example, if everything else remains the same, how far away would Station 2 have to be in order for the cost to break even? In other words, under what conditions does $S=T$? That is, for what values does $(P_1 - P_2) * n = (P_2 * 2m) / f$? If we use known values for all of the variables but the distance, m , we can answer that question. Team A might write the following:

$$(P_1 - P_2) \cdot n = (P_2 \cdot 2m) / f,$$

$$(\$3.59 - \$3.44) \cdot 15 \text{ gallons} = (\$3.44 \cdot 2 \cdot m) / (35 \text{ miles/gallon}),$$

$$\rightarrow m = 11.45 \text{ miles}$$

meaning that for the assumed prices of gas, fuel economy, and number of gallons to be purchased, if Gas Station 2 is less than 11.5 miles away, it is worth the trip, but if it is more than 11.5 miles away, it is better to just pay the higher price at the nearby station.

Another student wonders if it would make more sense to think about how big the price differential would have to be in order to make it worth it to drive to Station 2. Ultimately, this means using known values for all of the variables except for P_2 in the same equation, $(P_1 - P_2) \cdot n = (P_2 \cdot 2m) / f$, and then solving for P_2 .

Another student posits that they can look up the price of gas and they can look up the distance between gas stations. Maybe it makes more sense to see how many gallons of gas one would have to buy in order to make it worth driving across town for gas. This is just another rearrangement of the same equation.

After this rich discussion, Team A can finally see the power of this modeling approach. Their model allows them to not only answer the question of “is it worth it” for the current situation, but allows them to really get at the broader questions about that “tipping point” that would cause a person to go off of their planned route to purchase gas.

**ANALYSIS AND
MODEL
ASSESSMENT**

Team A now reflects on whether their model does a good job of answering the original question. Here is a list of a few of the questions they ask themselves:

- Is the sign of our answer right? (For example, students would want to save a positive amount of money?)
- Is our answer off by orders of magnitude? (For example, students would be concerned if they found they needed to purchase a million gallons of gas in order to make it worth the drive.)
- Does our model behave appropriately when we increase and decrease the input variables? (For example, if we keep all else constant, but then decrease the value m , the number of miles between the stations, do we see a decrease in T , the cost to travel the added miles?)
- Are our assumptions reasonable and defensible?
- Are our assumptions relevant?
- Does our model follow from our assumptions? Is our model completely explained by our assumptions, or do we need to add more assumptions?

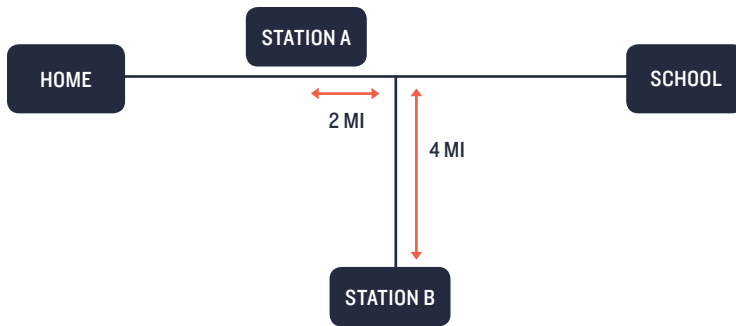


FIGURE 1.3: WHY TEAM A'S INITIAL ASSUMPTION ABOUT ADDED DISTANCE SHOULD BE REVISITED

- What if our assumptions are wrong? How does that impact our answer?
- What if our scenario changes a little? Do our results change a little or a lot?

One member of the team starts reflecting on the assumption they made about the way the team determined the added distance to go to the further gas station. She knows the locations of the gas stations and draws a picture, as shown in Figure 1.3, demonstrating that the added distance is much smaller than what the team is currently using. They all come to the conclusion that the distance between the gas stations is not quite the right thing to measure. Their initial assumption accounted for the total distance, 6 miles, between Station A and Station B. For the specific instance shown in Figure 1.3, they need only account for the 4 miles that Station B is from the route between Home and School.

They revise their assumptions and model to reflect this change, revisiting the model-building components, creating another iteration of their model.

REPORTING THE RESULTS

Team A feels confident in their revised model and is preparing to report on their results. Their report will use symbols and diagrams when necessary to explain their work. The symbols and equations will be intimately tied to the process that led to them, and would be virtually meaningless without explanation, so they know their narrative is also very important. The team uses this opportunity to defend the assumptions that led to their model in order to instill confidence in their model.

One team member discusses the project with her mother, whose gut reaction was, “There is no way I am driving 12 miles out of my way to save \$1.07. That is just not worth my time.” The value of a person’s time is something the team did not even think about when they came up with their solution. In fact, Team A’s model would suggest that driving out of the way is worth it even when the savings is only one cent.

The team member is panicked and contacts her teammates, worried that they have failed in answering the question, and they have to turn in a report the next day. Another teammate reminds her that every model has strengths and weaknesses. What is important is that they identify those strengths and weaknesses in their report. Those weaknesses can be included as possible future directions for their work.

All of this discussion about whether the team has really answered the “is it worth it” question inspires a thought in another student, who recognized that they did not account for any environmental considerations. Should they include anything about whether it is environmentally responsible to drive an extra 12 miles just to save \$1.07? Again, the team agrees that their model could be revised to account for environmental costs such as carbon offsets, and that could be listed in future work.

Note that with questions that are open-ended, students may even come up with completely different models.

We have now seen how the modeling process works by way of one example of how one team tackled one problem. Keep in mind that we cannot use this example as a template because every question inspires a different line of thought and every student or group of students will bring different ideas, background skills, and information about the problem into the modeling process. As modeling problems go, this problem was still fairly structured. For example, the problem statement indicated that gas prices and the distance to the

stations matter. Team A was able to develop a reasonable solution while considering many of the important factors. In Appendix C, we revisit this same problem to see other approaches.

Note that with questions that are open-ended, students may even come up with completely different models, all of which are valid yet result in different answers. As such, the assumptions and thought processes that lead to a model are every bit as important as the actual results.

**GUIDING
PRINCIPLES**

The following chapters will describe best practices for teachers supporting mathematical modeling. At all grade levels, the teacher has multiple roles, including:

- Selecting and/or developing modeling problems,
- Anticipating how students will attempt to answer the question,
- Developing a facilitation approach accordingly,
- Organizing and launching the students, and
- Providing students encouragement and guidance.

We close this introductory chapter by offering a set of teaching principles for the journey of implementing mathematical modeling in the classroom.

MODELING (LIKE REAL LIFE) IS OPEN-ENDED AND MESSY.

It may seem like a good idea to help students by distilling a problem so they can immediately see which are the important factors to be considered. However, doing so prevents them from doing this on their own and robs them of the feelings of investment and accomplishment in their work.

WHEN STUDENTS ARE MODELING, THEY MUST BE MAKING GENUINE CHOICES.

The best problems involve making decisions about things that matter to the students, and help them see how using mathematics can help them make good decisions.

START BIG, START SMALL, JUST START.

After reading this report, you may feel ready to jump in and make big changes, and if so, that is great! However, even small changes to things you already do in your classroom can encourage students to engage in mathematical modeling.

ASSESSMENT SHOULD FOCUS ON THE PROCESS, NOT THE PRODUCT.

Mathematical models (and the results they produce) are intimately tied to the assumptions made in creating the models. Assessment should be in service of helping students improve their ability to model, which will, in time, translate to a better product.

MODELING DOES NOT HAPPEN IN ISOLATION.

Whether students are working in teams, sharing ideas with the whole class, or going online to do research or collect data, modeling is not about working in a vacuum. The problems are challenging, and it helps to know you have support as you seek answers.

2

MATHEMATICAL MODELING IN THE EARLY AND MIDDLE GRADES: PREKINDERGARTEN THROUGH GRADE 8

INTRODUCTION

Mathematical modeling is a process, as described in Chapter 1. The focus of this chapter is how that process unfolds with learners in prekindergarten through grade 8. A look at how mathematical modeling can fit well in these grades, followed by ideas for facilitating and assessing students' modeling work, ends with awareness of how the guiding principles inform work with learners in the early grades.

The five Guiding Principles from Chapter 1 resonate with both effective teaching practices and productive dispositions as described in National Council of Teachers of Mathematics (NCTM) (2014) *Principles to Actions*¹. The practice of mathematical modeling is consistent with the CCSS.Math.Practice.MP4 Model with Mathematics, the other CCSSM mathematical practices as well as the literacy standards². Of course, individual mathematics content standards can be met, but this will be very task-dependent. Similarly, the P21 21st century skills of creativity and innovation, critical thinking and problem solving, communication and collaboration³ are all accessible via modeling.

Many of us have heard students ask “when are we going to use this?” in reference to their mathematics work. Mathematical modeling, because it tackles big, messy, realistic problems, helps students connect mathematics to life and empowers them to use their mathematics to solve relevant problems. Ideally we can find problems that are important to our students, such as community-based or school-based problems.

To understand how students grow in their ability to do mathematical modeling is a challenging task. Useful studies of mathematical modeling have been published but this area has not yet been researched as deeply and systematically as other areas, such as counting, whole number operations, fractions, and proportional reasoning. At this time, we can blend findings from existing research, expectations expressed in curriculum standards, and experiences from numerous classrooms to suggest general ways in which student mathematical modelers might progress from prekindergarten through grade 8 as they practice components of the modeling process.

**ADVANTAGES OF
MATHEMATICAL
MODELING IN
PREKINDERGARTEN
THROUGH GRADE 8**

Prekindergarten through grade 8 provides a natural setting for mathematical modeling with many advantages, such as harnessing and preserving the positive and inquisitive disposition of children and helping teach and assess learning of mathematics, other curricular content, and extracurricular ideas. Each of these is addressed below.

**HARNESSING
AND PRESERVING
CURIOSITY**

STUDENTS ARE ACCUSTOMED TO BEING CREATIVE.

Children are amazing! Their curiosity and fresh approaches to the world grab our attention. Their willingness to explore their surroundings, ask questions, and try new things—including things we might not have imagined — provides a perfect opportunity to engage in mathematical modeling. The open-ended nature of mathematical modeling problems elicits student creativity and choice⁴. Some students may not think of mathematics as a subject in which they can be creative. We hope that by engaging students in more creative endeavors during mathematics time we can widen the pool of students who continue their mathematics training.

THE DISPOSITION TOWARD AND MOTIVATION FOR MATHEMATICS IS HIGH IN THE EARLY GRADES.

Modeling in prekindergarten through grade 8 is one way to develop and maintain students' positive disposition toward mathematics. Children often develop the misconception that a speedy solution is best and that every problem has exactly one correct answer⁵. Modeling activities can help students avoid or dispel such unproductive ideas about mathematics, promote persistence in problem solving, increase comfort with multiple valid answers⁶, and provide students with more sophisticated metrics to evaluate the validity and usefulness of problem-solving strategies and solutions. We firmly believe that by engaging students in modeling from the earliest grades, they will grow to view mathematics as an engaging, fundamental, and useful way to approach life's critical problems and decisions.

EARLY AND MIDDLE GRADES STUDENTS HAVE POTENTIAL FOR FLUENCY AS SPEAKERS, THINKERS, AND DREAMERS OF MATHEMATICS.

We know that young children who are exposed to languages can readily learn them. We

would like to see children have similar exposure to mathematics with opportunities to develop mathematical fluency. To facilitate mathematical language development, we need students to engage in communication with each other and with fluent speakers of mathematics. As teachers, we can strive to continually deepen our own mathematical fluency and refine our practice so that we can maintain high expectations for our students⁷.

IMPROVED
TEACHING AND
LEARNING OF
MATHEMATICS,
OTHER SUBJECTS,
AND MODELING
ITSELF

MODELING CAN MOTIVATE MATHEMATICAL PROCESSES AND TOOLS.

Teachers have flexibility about how much they will point their modelers to particular mathematical tools. For a very open problem, students may realize they need a new tool such as multiplication as a more efficient way to conduct repeated addition. Sometimes students may choose a tool that is familiar to them and shy away from newer or more challenging ideas. In this case a teacher might allow a team to use the familiar tool and then ask the students to revisit the problem using a newer tool to see if they get a similar result and to compare strategies.

MODELING PROMOTES INTERDISCIPLINARY THINKING EARLY.

Taking advantage of students being with one teacher for different subjects or being taught by a team of teachers, lessons can address several subjects simultaneously through modeling activities. In many settings, early grade teachers work with their classes not only in mathematics but also in other academic and non-academic areas. Teachers can launch mathematical modeling problems from questions that children naturally ask about science, social studies, literacy, art, physical education, or music, which may occur during classroom activities or more casually in other settings. While this idea may seem to most naturally align to the sciences, such as using mathematical tools to help analyze data from science experiments, mathematical modeling can also tie mathematical tools to the humanities. For example, as students in the prekindergarten through grade 2 band develop reading skills, they might be asked to build a network model of three-letter words with two words

Teachers can launch mathematical modeling problems from questions that children naturally ask about science, social studies, literacy, art, physical education, or music.

connected if they differ by one letter, such as the one shown in Figure 2.1. Incorporating mathematical modeling — specifically by encouraging children to explore naturally arising quantitative questions using previously acquired mathematics tools — can strengthen their understanding of academic material and simultaneously reinforce the need for math.

For example, the book, *Misty of Chincoteague*, by Marguerite Henry tells the story of two children who work to own their own horses. The novel is set on Chincoteague Island, Virginia, which is famous for its herd of approximately 150 ponies that has been maintained by the local fire company for over 90 years. In the spirit of encouraging students to be

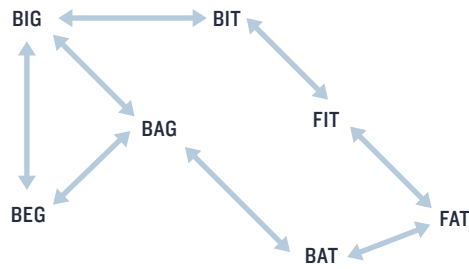


FIGURE 2.1: A NETWORK WITH A SELECTION OF THREE-LETTER WORDS AS NODES AND LINKS BETWEEN WORDS THAT DIFFER BY ONE LETTER

thoughtful, inquiring readers, one can encourage student questions like the following: Why are there 150 ponies on the island? How much water do the 150 ponies drink in a week? How much land do they need? How many more ponies could fit on the island? Students can answer these questions using maps, manipulatives, symbolic calculations, and other strategies of their own choosing. To simplify the problem enough to tackle it, they may need to make assumptions about the island and its populations. Approaching modeling tasks in such ways, students can connect their mathematical work and tools to the other disciplines, and see mathematics in their environment from a variety of perspectives.

WORK IN THE EARLY AND MIDDLE GRADES CAN LAY THE GROUNDWORK FOR MORE SOPHISTICATED MODELING.

The ways students engage in mathematical modeling in prekindergarten through grade 8 provide a basis for more sophisticated modeling processes. For example, if students have experience identifying and making assumptions about real-world situations during their first few years in school, they will be aware that solving real problems involves making assumptions when they encounter modeling problems in later grades. Early exposure to ambiguous or open-ended problems can help children become comfortable with the idea that useful, informative solutions to real problems are neither perfect nor unique. Students can develop a mathematical modeling disposition and competence in individual aspects of the modeling process and later combine these aspects into a complete, iterative process.

**IMPROVED
TEACHING AND
LEARNING OF
OTHER SKILLS**

MATHEMATICAL MODELING PROVIDES OPPORTUNITIES FOR COMMUNICATION DEVELOPMENT.

Through modeling activities all students can share, capitalize on, and develop their communication skills. Modeling problems are often posed in words and diagrams, and require students to express their answers using visualizations, abstracted mathematics, and prose. In addition, mathematical modeling teams must communicate well in order to succeed, and teachers can reach multiple teaching goals with one activity. The emphasis on communication may require special attention to students with language challenges, including English language learners. At the same time, because there are multiple avenues

for expression and because the problems have real-world context, modeling can provide positive experiences for students who might need to express their ideas in alternate ways.

MATHEMATICAL MODELING PROMOTES TEAMWORK.

Mathematics is sometimes seen as a solitary activity, perhaps reinforced by our evaluation of individual efforts in school and in competitions. Modeling is an inherently team sport and the problems are big and messy enough that a team approach helps students find useful solutions⁸. The job skills of work distribution, communication (including listening), and cooperation can all come into play naturally as a group works together toward their solution.

**IMPROVED
ASSESSMENT OF
LEARNING**

MODELING TASKS CAN HELP TEACHERS IDENTIFY STUDENT MISCONCEPTIONS IN A CONTEXT THAT IS FAMILIAR TO THE STUDENT, SO THAT THEY ARE BETTER ABLE TO EXPLAIN THEIR THINKING.

When students present their models, teachers and other students can comment on things that they notice and things about which they wonder. Invariably, presentations will include errors and opportunities for revision. Teachers (and classmates) can determine the nature of the errors by asking students to explain their thinking.

As a reader of GAIMME, you might already be convinced that prekindergarten through grade 8 is a terrific setting for mathematical modeling. However, in the event that you were skeptical, we hope that the ideas above convince you that not only are there many benefits to starting early, but also that the prekindergarten through grade 8 classroom situation and timing has some distinct advantages. In the next section, we turn to how prekindergarten through grade 8 students can experience modeling, building up to a full modeling process by starting from its components.

**ENACTING
MODELING
COMPONENTS AND
PROCESS WITH
STUDENTS**

Throughout this section we pay particular attention to components of the modeling process and how students engage in modeling within the grade bands as defined by the NCTM: prekindergarten to grade 2, grades 3 through 5, and grades 6 through 8. Engaging in a whole iterative modeling cycle as mature mathematical modelers is the goal. Beginning modelers can focus on a single component, such as making assumptions or assessing the pros and cons of a particular model.

We consider the components with early and middle grades students in mind. For each grade band, we describe what might happen using the modeling components in “Getting Started & Getting Solutions”⁹. We first discuss each of the modeling process components through the lens of an open ended question: “what should you bring for lunch?” We then turn to examples of how modeling work inspired by this Lunch problem might unfold for each of the grade bands. Among other things, the use of this common starting question illustrates that a single modeling scenario can lend itself to many different modeling problems and be visited and revisited at different grade levels.

**START WITH A REAL
WORLD PROBLEM**

Genuine mathematical modeling problems start out big, messy and unmanageable. Modelers have to decide the specific problem they want to solve, what information they will need to solve it, what mathematical tools will be useful, and how they will know when they have a useful solution. This is very different from traditional word problems or textbook applications where all of the necessary information is provided and there is a single, known, correct answer.

To facilitate mathematical modeling, the teacher first chooses a scenario — such as the Lunch problem context — that is relatable for their students and has questions that someone cares about. That person might be the cafeteria manager, a parent or caregiver, a local businessperson, or someone else at their school. To engage students, the problem can be posed as a story where an explanation, decision or strategy is required. If the story is open-ended enough, it encourages the modelers to ask questions that will help them decide how to approach a model and solution. As students get older they can participate in the selection and development of problems. Usually starting with a rich scenario, teachers and students will have to refine the question that they will use modeling to answer. For the Lunch problem, they might have to decide they are bringing a lunch on a typical school day and not for a special event or holiday.

**MAKE
ASSUMPTIONS AND
DEFINE ESSENTIAL
VARIABLES**

Once students (perhaps with facilitation from their teacher) have decided what specific problem to approach, they can think about what information they will need to solve the problem. In simple terms, they need to figure out “what matters.” To start, teachers might ask students to say what matters to them without formally noting that the answers could lead to assumptions in modeling. For the Lunch problem, even responses as general as “things I like,” “healthy stuff,” and “enough to fill my lunch box” can be good starting points. In addition to generating individual assumptions, students may consider pairs of assumptions and discuss the compatibility (or incompatibility) of their choices. Can we include both “things we like” and “healthy stuff?” Students can be encouraged to move beyond simple comparisons of assumptions to discussing their implications in the context of the problem. We like bananas and they are healthy for us, but do they fit in our lunchboxes? They could also justify why some quantities should be fixed while others should vary. If some information required for the model is not available, students can estimate values for quantities and justify those assumptions. How big is a banana?

Another possibility is to give students a model and have the students discuss what assumptions are made in the model. One Lunch problem solution might include one medium orange, apple, or pear. Students could surmise that the model assumes each lunch must include a serving of fruit and these fruit travel well or are commonly liked. Poking to see what assumptions seem to underlie a model provides a way to see assumptions functioning in the context of a model even before students are ready to generate models themselves.

Taking this exercise one step further, student could think about how the model and its solutions might change if different assumptions had been made. What happens if we assume the total cost of the items we pack has to be less than \$5, and then change it to less than \$3? Later, this analysis of the assumptions will be important when students take a critical look at their own models and solutions.

**USE MATHEMATICS
TO GET A SOLUTION**

An important characteristic of mathematical modeling is that students not only make choices about how to narrow the focus of a problem but also are empowered to choose a mathematical approach. In the early grades, making and implementing decisions about how to use mathematics to get a solution might happen in a group setting. Some young children might benefit from use of visual representations to help them recognize a pattern or trend while others might recognize the pattern something familiar to them. For example, the number of carrots in a lunch might be 0, 6, or 12, which some young students might recognize as multiples of 6 and later learn that is the number of carrots in a standard serving. Students could begin to form individual conclusions or work in small groups to consider multiple solutions to a particular problem and begin to compare/contrast the solutions.

Students should become increasingly able to select and apply more sophisticated mathematical tools and visual representations to develop and solve their models. They also should connect their choice of tools to assumptions and variables. For example, an assumption in the Lunch problem might be that the meal needs to be nutritious. Grade 3 students might initially consider nutrition in terms of the whole number of calories for students. Grade 6 students might consider nutrition as the percent of of recommended daily requirement of particular nutrients, and then draw on their ability to reason proportionally. Teachers can encourage teams to consider more than one strategy and to search for additional mathematical tools before they settle on any particular approach and tools.

**ANALYZE AND
ASSESS THE MODEL
AND SOLUTION**

Because all models involve assumptions and approximations, a critical aspect of mathematical modeling involves assessing the ability of the solution to satisfy the problem. Various questions about the model might be useful in helping students to learn to analyze and assess models and solutions. In what contexts does the model provide valuable information? How precise are the solutions? How much do the solutions change if you slightly vary the assumptions, model, or parameters? Do the results make sense? Would it improve the model to change something decided at an earlier step and resolve the problem? Questions for the Lunch problem might be as different as “what if we change from a lunch box to a lunch bag” or “we wanted to fill our lunch boxes – does our model ever produce lunches that overflow the boxes?”

**ITERATE AS
NEEDED TO REFINE
AND EXTEND THE
MODEL**

Models are never perfect because they are simplifications of reality. However, the simplicity of a model can be a positive quality because, for example, it can show trends or cause and effect. At the same time, models will not hold for every situation or represent a real situation exactly. Part of the modeling process is a continual evaluation of the limitations and sources of error in a model based on available information. Ideally, students at all levels will have the opportunity to revise and improve their models based on new discoveries made during the analysis component. Students working on the Lunch problem might find that their model works well until they discover they have missed a vital point, such as they did not include a beverage, or they change an assumption, such as how many hours will pass from the time the lunch is packed until it is eaten. Sometimes iteration cannot happen due to time constraints, but even a discussion about possible improvements conveys to students that models are rarely static, perfect solutions to a problem.

Iterations of the modeling process can provide opportunities to motivate and introduce new content.

Further iterations of the modeling process can provide opportunities to motivate and introduce new content. A first model may apply previously-learned material, but while discussing possible improvements to the model, it may become clear that the class does yet have the mathematical equipment to improve it as desired. A decision in the Lunch problem to consider of total fat as well as total calories

move students from working with linear equations to a system of linear inequalities, and cause students to use laborious methods or change their plans. Such incidents can motivate students and provide valuable opportunities to introduce, explore, and apply new mathematical tools.

**IMPLEMENT THE
MODEL AND
REPORT RESULTS**

Modelers could iterate and improve indefinitely, but deadlines happen. In school and on the job, at some point it is time to declare victory. The good news is that when students report their results, they can explain what they would do to improve their model and in what situations the model would or would not be useful. Substantial work on the Lunch problem might produce a defensible model but note that the result does not take into consideration a range of food allergies or accounts for the cost of the meal but not savings that might occur if food is bought in bulk.

Students have many ways to present their results. Younger students may wish to show and play act to demonstrate their ideas — or with a packed lunch box. With help they can display data they have collected in some kind of chart or diagram. They could also choose to use manipulatives to communicate or draw simple representations of mathematical ideas, such as showing how the relative length of a banana and width of lunch box are related. Very young children can make and report comparisons. As students develop their language skills, they can make more formal presentations, both in terms of sentences and mathematical representations and visualizations. As students begin to think more abstractly, they can

generalize and argue their claims using discussions, reports and posters. Modeling provides opportunities for students to transfer mathematical communication and representation skills to new contexts, and can allow teachers to assess whether students do so appropriately.

As in other situations that require presentation and discussion of mathematical ideas, teachers can support both English language learners and students with expressive language challenges with alternative or supplementary ways to express their ideas. For example, students who are struggling to produce the words needed to describe their assumptions might draw pictures to illustrate each of their assumptions or to initially draft their ideas in their first languages.

**COMPONENTS AS
PART OF ITERATIVE
PROCESS**

To this point, we have considered how individual components of the modeling process might play out with learners in prekindergarten through grade 8. Students also need to understand how these components come together within a complete modelling activity. As illustrated in Figure 2.2, the components do not usually unfold in a simple linear way. For example, in this figure, after students identify the problem and assumptions, they loop through finding and analyzing a solution twice before moving on to refine the model and report their results.

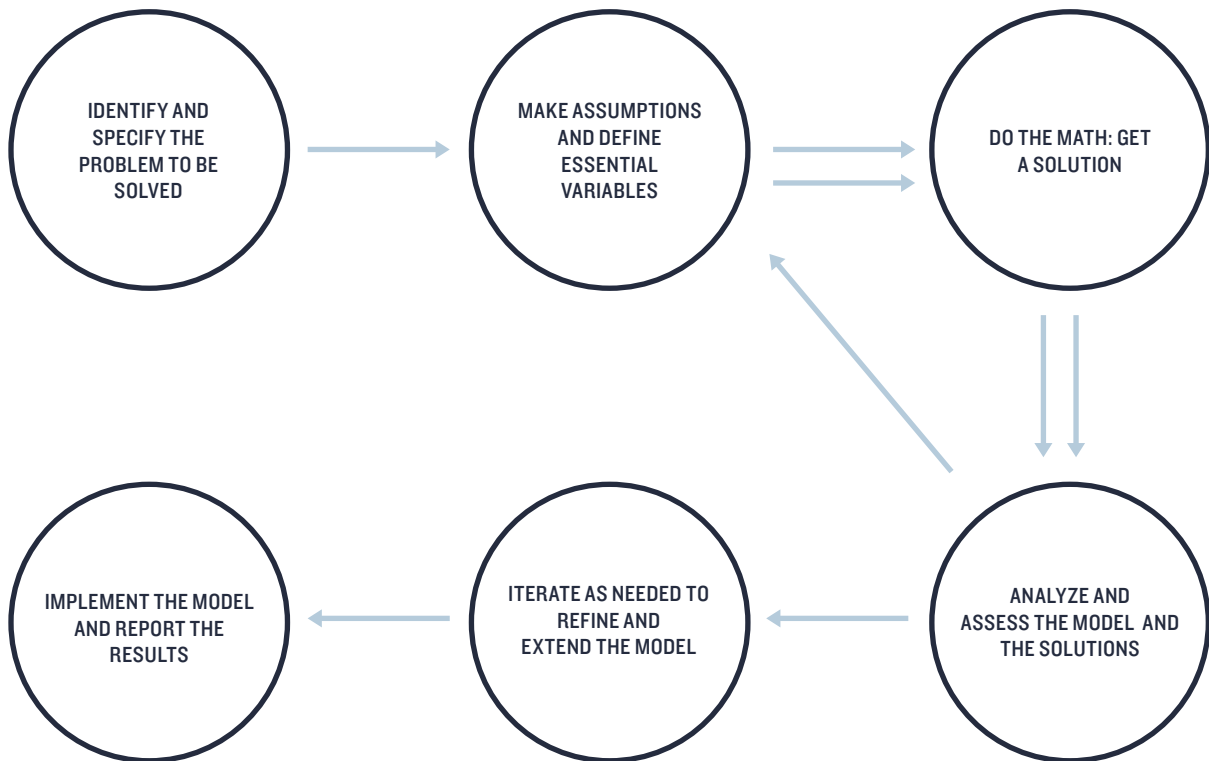


FIGURE 2.2: A SINGLE LOOP BACK AS ONE EXAMPLE OF A NON-LINEAR PASS THROUGH A MODELING PROCESS.

We end this section with examples based in the Lunch problem to illustrate how the full process might play out within each grade level. In the next section we move into a practical discussion of how to plan, launch and facilitate mathematical modeling lessons in the early and middle grades.

Lunch Problems: Grades prekindergarten through grade 2. In prekindergarten through Grade 2, the class likely will focus on a central question, which could be motivated by student interest, but then is identified, selected, or articulated by the teacher. For example, to focus the discussion in the Lunch problem, the teacher could have students discuss a food that they all eat, such as carrots. Once the decision has been made to think about carrots, the modeling problem can be expressed in the form of a focusing question. The question could be quantitative, such as “how many carrots should be in the lunch,” or qualitative, such as “which foods go best with carrots?” Either way, the goal is to use mathematics to answer the question.

The question about how many carrots should be in the lunch is intentionally open. To determine a reasonable answer to the question “how many carrots should be in the lunch,” students might collect data from their lunches. They might count the number of carrots in their lunches each day in a particular week. Very young students might be able, for example, to count and place the same number of stickers for a histogram as the number of carrots in their group’s lunches. Graphing can be done as a whole class until students are ready to try it independently. A final important step is to move beyond recording the data to deciding what information or conclusions can be drawn from the visual representations of information. The class might even decide that they want to collect more information about carrots or about a different lunch item.

To explore a qualitative preference such as how much students like something, the class could develop a scale to represent how much they like it. The scale could be pictorial (such as the various faces along a scale used by medical professionals for children to show how much pain they are in) or numerical (building upon students’ understanding of number lines). Preferences determined by using these data and illustrations could be used by the class to inform decisions such as what snack a parent will bring for the next class event.

Prekindergarten through grade 2 teachers can capitalize on young children’s ability to act out a situation, count, compare size, and make 1-1 correspondences. Another approach to the Lunchbox problem could focus on how many carrots can fit in a lunchbox of a particular size. In the early grades, students could create experiences to evaluate and compare results from their models. For example, students could use blocks of appropriate sizes to represent different food items in order to investigate how to pack lunches in different size lunchboxes. They could then determine and compare how many items and which combinations of items fit into each box. As they analyze their answers, they could make guesses about how their

solutions would change if the lunchbox was twice as long.

It is possible but not essential to use the vocabulary of modeling such as “making assumptions” as students work on a problem. In these early modeling experiences, the big idea is to show young students how to frame, explore, and answer relevant real world questions using mathematical ideas.

Lunch Problems: Grades 3–5. In grades 3–5, students can play a larger role in generating and defining the big modeling questions they would like to address. The prekindergarten through grade 2 students likely would answer the question about how many carrots should be in lunch based on the concrete experience of what they see in their own lunches. In grades 3–5, students could take more sophisticated information into account, such as nutritional information, quantities, taste preferences and variety. For example, they might be able to make arguments about nutrition to compare carrots to another choice of vegetable.

Students could also consider a larger question, such as how to compose the best lunch from a variety of food choices. To make the problem more manageable but still open, the teacher could introduce the number or type of items in a lunch as a basic assumption by taking a recent weekly or monthly lunch menu and using those as the possible choices. The general question then becomes “what is the ‘best’ lunch?” Presenting solutions then involves the mathematical definition of the word ‘best’ and justification of that approach, including pros and cons of the different definitions. For example, an answer that allows many different combinations of foods might not be feasible for the cafeteria, which can only produce a certain number of meal choices each day. An answer that produces the healthiest lunch might suggest lunches that are too expensive or are not considered tasty by some students.

Students can also think about extreme cases and whether such cases can be captured by the model. For example, students could be asked what changes if they are filling a cafeteria tray rather than packing a lunchbox. This “what if” type thinking can lead to interesting analysis of the model and a broader sense of the kinds of thinking – such hypothetical, conditional, and logical thinking – that are a part of mathematics.

In grades 3–5 teachers can encourage students to consider the predictive value of their models. For example, how many carrots should the cafeteria order each week? As students learn about place value, they should be able to notice whether numbers are of reasonable magnitude – a total of 100 carrots for an entire school might not make sense! Estimation could come into play, and students may try to quantify the possible sources of error or uncertainty in a model. For example, if we round the cost of one meal from \$3.89 to \$4.00, a difference of 11 cents, the annual cost of the meals at a school with 450 students would be off by more than \$8910. When students represent their answer using fractions/decimals or scientific notation, teachers can ask them to justify their choices for using

each representation.

Use of the vocabulary of modeling, such as “making assumptions,” as students work on a problem can facilitate conversation. In modeling experiences for grades 3–5, a big idea is to encourage students to move beyond relatively narrow mathematical descriptions of concrete situations to consider larger questions, combinations of assumptions and variables, extreme values, and predictive potential to construct and challenge the models they build.

Lunch Problems: Grades 6–8. In grades 6–8, students can be considerably independent. The class as a whole might be considering a big question, but each group of 3 to 5 students might look at a particular aspect of the problem that interests them. In comparison to their work in earlier grades, students can likely make more sophisticated arguments, simultaneously taking into consideration several constraints such as space, preferences, and nutrition.

As with the younger students it might be helpful to use a monthly lunch menu in order to identify (and justify) standard components of a lunch. Students could study different voting systems to determine what kind of pizza they might order for a class party to make the most people happy. Different groups could make different choices such as what would be the pizza company, delivery distance, cost, delivery options, gluten-free options, toppings, and crust type. The assumptions part of the modeling process could involve limiting these options to a reasonable number and possibly even limiting the choices within a particular option, such as which toppings to consider.

Students should be able to begin to use variables to represent quantities that change and use equations to represent relationships between the quantities. Even with qualitative questions such as what is the best pizza (or combination of pizzas) for the class to order, students can devise a scoring system that takes different attributes and weights into account. They could have a function for the cost of a pizza that incorporates not only pizza choices, such as size and toppings, but also gas to pick up the pizza or time versus delivery cost.

Students in grades 6–8 should be able to provide multiple representations of their results including symbolic, graphical, and verbal (written and spoken). As students report their results, especially when students have some experience with modeling, they should be expected to be critical of their own work. Teachers can ask students to discuss pros and cons of the choices they made, both in the assumptions component and in the choice of mathematical technique. Mature modelers may demonstrate and discuss what happens to their solution when they change an assumption or a particular number (a process known as sensitivity analysis because it shows how sensitive the model output is to changes in various inputs). For example, if one of the parameters is the number of apple slices, you could ask what might change if the number is increased or decreased” — does it make the meal

more or less appealing per class vote, or how does it affect calories in particular or nutrition in general.

The vocabulary of modeling, such as “making assumptions,” can facilitate conversation. A big idea in grades 6–8 is to encourage students to act more independently as they tackle bigger problems with a growing collection of mathematical tools. The Lunchbox problem and its possibilities have a nice connection to one of the Moody’s Mega Math Challenge problems (<https://m3challenge.siam.org/archives/2014/problem>). The problem — Lunch Crunch: Can Nutritious Be Affordable and Delicious? — asks how to make lunch simultaneously delicious, nutritious, and affordable. After working on their aspect of the Lunchbox problem, middle grades students might like seeing that a similar problem was posed in a national modeling competition for high school students. They also can see examples of winning solutions on the Moody’s website.

**FACILITATING
MATHEMATICAL
MODELING
LESSONS**

Students in prekindergarten through grade 8 are ready in many ways to engage in aspects of mathematical modeling. Because modeling is usually engaged as a group activity, student groups may be working semi-independently (and more so as they mature). A key to facilitating mathematical modeling with students is syncing the teacher’s role with the students’ mathematical modeling process. In mathematical modeling, as with any lesson, the role of the teacher before and during the activity matters.

**HOW DO
STUDENTS IN
PREKINDERGARTEN
THROUGH GRADE 8
DO MATHEMATICAL
MODELING?**

As you may imagine, students at various developmental stages will engage with mathematical modeling processes differently. Research evidence¹⁰ indicates students in the early grades can engage, at least to some degree, in the same components of the modeling process as students in higher grades. In our experience, young children throw out a lot of (sometimes wild) options and need help in focusing while older students are likely to give a standard reply and need help in expanding their thinking.

Teachers will need to carefully facilitate the modeling process while maintaining open-endedness in the problems and genuine choices for the students, as developmentally appropriate. Despite developmental differences, students at all grade levels can make meaningful choices that shape their modeling experiences.

Below we will describe the actions of the teacher as a modeling facilitator. As you read this section, it may be helpful to keep some example modeling problems in mind. Appendix B describes how students within each NCTM grade band might go through a modeling process to address the same very open-ended question: What should you bring for lunch? We refer to this problem as The Lunchbox Problem. Appendix B explores this problem with a class of first grade students, illustrating how children in the prekindergarten through grade 2

band could engage modeling with this problem. We also follow a group of 4th graders as they address this problem, and view a group of 7th grade students working on the problem. The successful experience of each group as they engage with modeling experience is shaped by their own choices, while the teacher facilitates. The next section provides a framework for teachers of modeling.

**WHAT ROLES DO
TEACHERS PLAY IN
SUPPORT OF
STUDENT
MODELING?**

Carlson, Wickstrom, Burroughs and Fulton¹¹ (2016) have developed a framework for teaching modeling in elementary grades (see Figure 2.3). The inner cyclic arrangement of three squares with rounded corners represent the students engaged in the three phases of a modeling cycle — Pose Questions, Build Solutions, and Validate Conclusions. This is a simpler modeling process diagram than the one described in chapter 1. The primary teaching actions are represented by the bold phrases: Developing & Anticipating, Enacting, and Revisiting. The cycle represented by the outer triangular cyclic arrangement represents three teacher actions — Organize, Monitor, and Regroup — to facilitate modeling as the students are engaged in Enacting.

You may see strong similarities with the five practices for orchestrating mathematical discourse by Stein and Smith¹², but you will also see some differences due to the open-endedness of modeling problems. For example, the order in which student groups present their modeling solutions is important. A teacher might choose to have a moderately successful solution presented first so that strengths and weaknesses can be identified, This can normalize the idea that even a “finished” model is never perfect. Another possibility is to order presentations by the type or specificity of assumptions made by the groups. This choice can be made according to what the teacher would like to emphasize.

Because mathematical modeling is an iterative process in which modelers loop back to improve their own solutions, the sequencing of solutions is something in which students take the lead when they report on their work.

**ANTICIPATING AND
DEVELOPING: THE
TEACHER ROLE
BEFORE MODELING
BEGINS**

As part of their planning, teachers anticipate by choosing/developing a modeling problem and imagining what might happen in the classroom. The added attention to facilitation is indicative of the usually longer, messier, and open-ended nature of modeling work and the importance of digging into the nuances of the modeling context.

DO I HAVE TO COME UP WITH MY OWN PROBLEMS?

Even experienced modelers can find the task of creating successful modeling problems challenging. Selecting and refining existing problems can free the teacher to focus on implementation. With experience teachers can become comfortable developing modeling problems and capitalize on events in the classroom, the cafeteria, and the playground to turn them into modeling problems that resonate with the students. Whether you are

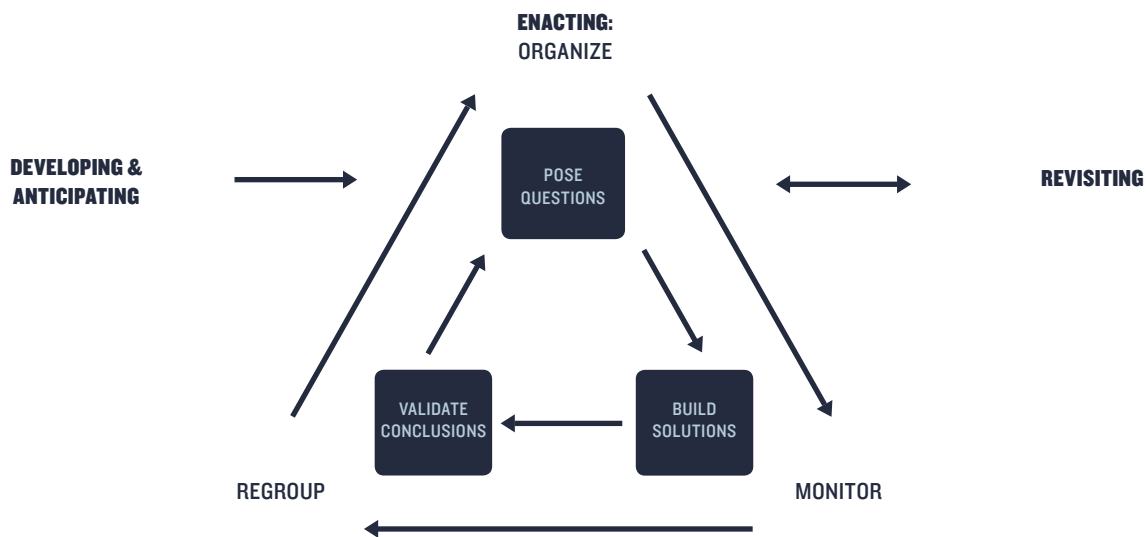


FIGURE 2.3: FRAMEWORK FOR TEACHERS' ROLES IN FACILITATING MODELING, INCLUDED WITH PERMISSION FROM CARLSON ET AL. (2016).

developing a new problem or adapting a modeling problem that someone else created, the following questions can help guide your choices.

IS THIS PROBLEM REALLY A MODELING PROBLEM, AND HOW CAN I LAUNCH THE PROBLEM SUCH THAT IT IS MESSY AND OPEN-ENDED?

As we noted in Chapter 1, simply adding labels or context does not make a problem a good modeling problem. A modeling problem should also involve student decision-making that shapes the problem definition, modeling processes and variety of the solutions.

WHAT MATHEMATICAL CONTENT DO I WANT TO ADDRESS IN THIS MODELING PROBLEM, AND DOES THIS PROBLEM ADDRESS THAT CONTENT?

Whether the modeling activity promotes curiosity and discovery in a new topic, provides an opportunity to put a recently acquired skill into practice, or to draws connections and revisits previously presented content, if you want the modeling process to lead towards specific mathematical content, you will want to develop the problem and/or facilitation with these learning objectives in mind. As shown in Chapter 1, modeling problems can be developed by adding flexibility and room for creativity to existing book problems. Appendix B offers two additional examples of modeling problems, both designed to target the specific modeling topic of division, although the two problems have different levels of open-endedness. The Buses and Quotients problem opens a door to modeling only by

asking students to consider the reasonableness of the numerical results of calculations. The Trapezoidal Teatime centrally engages students in writing algebraic expressions, which is useful in developing the model.

I WANT TO DEVELOP A MODELING QUESTION AROUND SOMETHING EXCITING THAT MY STUDENTS NOTICED. HOW MIGHT I TURN THAT INTO AN AUTHENTIC MATHEMATICAL MODELING EXPERIENCE?

Children are always observing the world around them and making conjectures. In some of these instances, you can transform these moments into mathematical modeling problems. For example, when a child looks at two books and comments that it will take longer to read one than the other, there is an opportunity to pursue the question of how long it will take to read a book as a modeling task. Young students might start with two specific books and offer that one book will take longer to read because it is heavier or taller than another. Their reactions are all part of their developmental ability to conserve and compare quantities. Older students may open the last page of two books and compare the total number of pages in both to make a choice. In this instance, it is reasonable to ask the student if they know of any additional characteristics besides page count that can factor into the amount of time it will take to read. Measuring the time it takes to read the books can help test and refine their thinking about what attributes to consider, how to quantify them, and how to refine their ideas. Students might then be asked directly to predict how long it will take them to read a book. This type of question can encourage students to start thinking about answers as a range of values rather than as a single number.

ONCE I HAVE A PROBLEM, WHAT ELSE DO I NEED TO ANTICIPATE?

Facilitating modeling often depends on timing – deciding how long to allow students to struggle so that they learn to persevere through difficulty, and when to regroup the class to provide new information or guidance. With a multi-day modeling task, things can be often be somewhat chaotic the first day. Then with time, groups usually can settle into a plan and enact it. The following questions can help anticipate the directions students might take:

- What kinds of questions will the students need to ask as they try to define and answer the problem?
- What information might they seek and from whence will they seek it?
- Which discussions should happen in small groups and which with the whole class?
- What decisions will students need to make and what information will they need to make them?
- What mathematical approaches might they take?
- Will there be pieces of information to introduce later if students need steering in a particular direction?
- How can I nudge my students without depriving them of the opportunity to develop concepts and discover relationships among them?
- How many days should students work on this problem? How long each day?

**ENACTING:
THE TEACHER ROLE
DURING MODELING—
ORGANIZING,
MONITORING, AND
REGROUPING**

The next step is enacting, in which teachers organize the students and launch the modeling problem, monitor students as they work, and periodically regroup them. The idea of this framework is to separate the work of the student and teacher. Rather than demonstrating modeling to the students, the teacher is more like a coach. In *organizing*, the teacher *launches* the problem by providing students with the basic rules of the game, such as general problem to be solved, some mathematical strategies and some ideas about how to work as a team. As the students engage in modeling, the teacher *monitors* the students. What approaches are they taking? What mathematical opportunities are arising? Where are they getting stuck? Then periodically the teacher can call a time out to *regroup* the students and address a misconception, answer clarifying questions or provide a small suggestion.

The iterative nature of modeling requires students to select which of their ideas to present as the current model, perhaps after they sequence their solutions and before they refine their models. In short, students as modelers are responsible for choosing their approaches and connecting their models to the broader context of the original task. Students take over the selecting, sequencing, and connecting, especially when engaged in the iterative work of building, assessing, and improving a model and producing a report. In summary, for a teacher, supporting modeling often means the role of the teacher is not to demonstrate the techniques but instead to encourage students to choose, develop and apply mathematical approaches.

The role of the teacher is not to demonstrate the techniques but instead to encourage students to choose, develop and apply mathematical approaches.

Once students develop a set of solutions, teachers can take the opportunity to help students think about how they might approach the problem if they were to engage in a second iteration of modeling to improve their answers. Describing the strengths and weaknesses of their model may be a standard requirement of the problem. Not every suggestion for improvement may be equally useful. For example, students might note that they can improve the precision of their model and can discuss at what point the numerical precision loses meaning in the real-world context.

As students work in groups, teachers can monitor their progress by walking around the room, listening to their conversations and occasionally stopping to ask them to explain their current thoughts. The goal of the monitoring process is to insure that students are heading down an appropriate path, not necessarily a single “correct” path. When students take an unexpected (and potentially not fruitful) path, it takes discipline to refrain from immediately redirecting their work by regrouping too soon. It can be valuable for students to discover when to redirect themselves toward a new approach.

In a regrouping, teachers may have teams report their progress followed by a whole-class

discussion. You may also provide some additional information or ask a probing question to focus the teams' efforts. You must decide when and what individual groups should share with the class, depending on how much you want the students to influence each others' approaches. Having less communication might lead to more interesting variation in the solutions, while more communication might lead the whole class to apply a particular tool or approach. Also, sharing information earlier in the process can lead to groups having more similar experiences than if they worked in isolation until later in the process. Therefore, the goals of the activity will determine the level of intervention.

**REVISITING: THE
TEACHER ROLE
AFTER A ROUND OF
MODELING**

Part of teaching modeling is having students revise and refine their solutions. It can also be useful to revisit a problem as a class, which means come back to take another look at the problem from a different perspective or with additional tools (Carlson, Wickstrom, Burroughs, and Fulton). Modeling problems can often be enacted over several days, which enables you to consider whether you want to have the class revisit a problem from another angle. Using the lunchbox problem as an example, maybe the first launch of the problem is "How many carrots should a lunch contain?" and then another day you and your class can revisit the problem by asking "How much drink should a lunch contain?" or "Pick a different vegetable than carrots, and use the nutrition facts (or one particular fact) to determine which vegetable is a better nutritional choice. A more sophisticated question could ask students to describe the pros and cons of each choice." In each case, the students justify their answers using mathematics.

WILL STUDENTS AUTOMATICALLY ENGAGE IN MATHEMATICS TO ADDRESS A MODELING PROBLEM?

One important question is whether you want students to use particular mathematics to address a modeling problem. The most authentic problems leave this choice up to the student, but sometimes it can be appropriate to remind students of some of the tools they might choose to employ. Students can brainstorm as a class to come up with approaches, or groups can try to think of more than one approach before they begin to solve the problem in earnest. It could be valuable to do some of this early brainstorming as a class, without necessarily working towards a solution, so students can see there are multiple ways to use mathematics to get started.

Sometimes as we develop problems, we might not anticipate that students will engage in a way that allows them to avoid using mathematics. Appendix B offers an example of how something originally conceived to be a good mathematical modeling problem failed to get the students to use mathematics because they spent more time on the design aspect of the problem than the mathematical model. The scenario clearly engaged students but the implementation allowed students to seem successful without having to bring mathematics to bear on the question. This apparent failure could provide an opportunity for the teacher

to revisit the problem and refocus the students on a mathematical approach.

It is important to recognize that modeling is not just an “end of the chapter” activity; problems can be launched when the precise mathematical equipment needed is unclear or unknown. An initial model might use content that students already know and serve to point out its inadequacy for the problem at hand. Revisiting provides a prime opportunity to motivate and introduce new mathematical material.

ASSESSMENT

As we have seen throughout this chapter, mathematical modeling can be used in the lower grades:

- To promote mathematical content through the motivation of new material, the implementation and practice of previously learned skills, or the synthesis and integration between mathematical topics;
- To introduce quantitative thinking in arenas other than mathematics, by having students use mathematical modeling to address questions that naturally arise at lunch, on the playground, during reading time, in science, or elsewhere;
- To develop positive aspects of students’ mathematical work, such as creativity, persistence, team work, quantitative reasoning, and communication, as they engage in modeling, through separate components or as an entire process.

Depending on your goals for a particular modeling exercise, you will probably want to assess different outcomes. Below we address some ideas of how to assess the learning of mathematical content, other skills, and modeling itself. For additional insights please see Appendix D.

ASSESSING MATHEMATICAL CONTENT

If your goal is to have students practice specific mathematical content, then by using a modeling problem, you can still assess the mathematical correctness in a familiar way. Furthermore, you can ask students to explain their answer in the context of the modeling question, which provides a more intimate view of how well students have internalized the underlying concepts.

ASSESSING SOFT SKILLS

Because modeling problems are often approached by teams, the question of assessment has many facets. It is possible to ask students to describe both their own contributions and that of each of their teammates. It is also possible to require older students to write reflections not only about the product of their modeling but also the process. Many of these skills needed to complete big messy mathematical modeling problems as a team are highly valued in the workplace and can be included in the grading process or rubric.

**ASSESSING
MATHEMATICAL
MODELING**

The activities pursuant to mathematical modeling are not necessarily aligned with more familiar teaching and learning experiences. When you first begin teaching mathematical modeling, it may be difficult to tell whether your students are actually engaged in the modeling process, let alone assess how well they are modeling. Figure 2.4 offers a list of questions that you can ask yourself throughout the modeling experience to assess whether students are participating in the mathematical modeling process.

Assessment should focus on the holistic process of mathematical modeling. As you assess student-developed models, you should not just check a final answer to see whether it matches an expected response. Assessment should have a primary focus on what pieces of a modeling process are present and how these pieces are logically connected. Quality work includes reflection about the result (and communication of the result), how the model could be improved, when it is useful, and how it is limited. Identification of mathematical misconceptions and mistakes should be a part of the revision and evaluation processes, but is just one of many important factors. Responding to modeling can be much more challenging than correcting a paper with a clear goal of one particular answer per problem. Meeting this challenge can provide students with meaningful feedback about many areas of their academic development.

In some cases, a student may develop an idea or technique that seems unfamiliar to the teacher. Most teachers of modeling (including the writers of this guide) have been in situations where they are not sure in the moment whether a particular approach is viable and correct. In this case, asking the student to explain their thinking and then taking some time to consult resources with information about the context. Considering whether student ideas are viable is a task that modeling facilitators at all levels often have to do. Moments of teacher uncertainty allow students to see that learning is truly a life-long endeavor and that adults are not infallible entities. Students can observe how their teacher continues to embrace new questions and actively question and seek answers. Teachers can demonstrate curiosity and persistence. This learning together of new facts, conventions, and approaches can be one of the most fun and rewarding parts of teaching mathematical modeling.

As with other areas of mathematics, formative assessment can provide a rich picture of students progress. Thus grading modeling is not just about checking off components completed or checking for mathematical accuracy. It is about understanding students' mathematical thinking and attending to how well their reasoning justifies the appropriateness of their model and resolution of the real world problem.

There are also many opportunities to assess students at each component of the modeling process. Appendix D offers some questions designed to assess the modeling process while providing a structure for meaningful feedback. The vocabulary is basic enough that students could be encouraged to use them for self-assessment, as well.

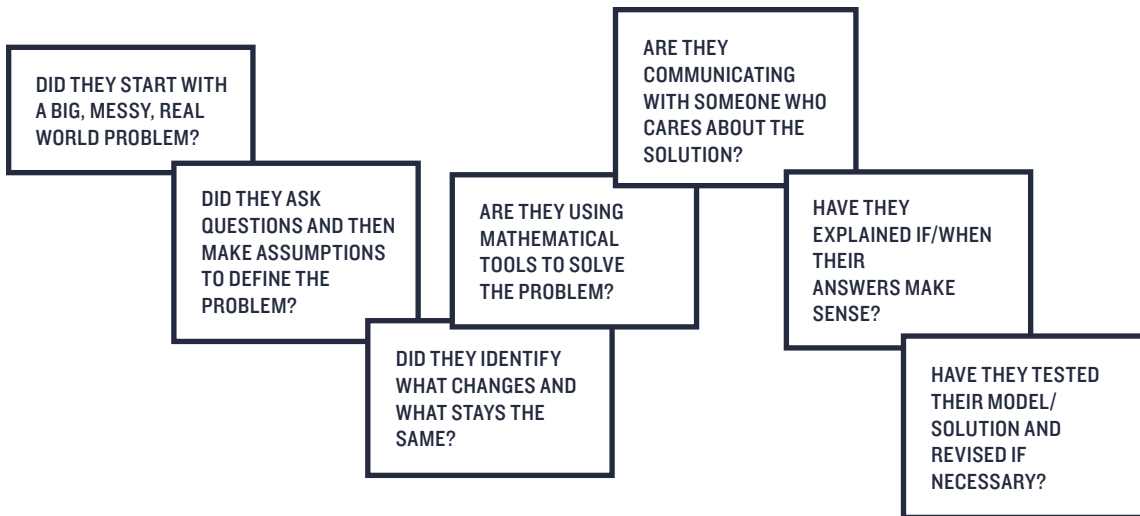


FIGURE 2.4: QUESTIONS TO ASSESS WHETHER STUDENTS ARE MODELING. USED WITH PERMISSION FROM LEVY, IMMERSION⁹

CONCLUSION Mathematical modeling in prekindergarten through grade 8 allows kids to use mathematics to pursue questions they naturally ask about the world. This section plus the examples in the appendices provide early disposition and skills for mathematical modeling that can be expanded and leveraged in future years. Kids can develop early disposition and skills for mathematical modeling that can be expanded and leveraged in future years.

3



MATHEMATICAL MODELING IN HIGH SCHOOL: GRADES 9 THROUGH 12

INTRODUCTION

As students move through the high school grades, the modeling process and the guiding principles for teaching mathematical modeling described for the early and middle grades remain intact, while the mathematical content of the high school curriculum offers the students many new tools appropriate for mathematical modeling, such as algebra, geometry, precalculus, and statistics.

Simultaneously, the life experiences and interests of high school age students are rapidly expanding, opening the door to a vast array of real-world problems for their investigation.

Mathematical modeling in the high school setting can take many forms. Models can be used as motivation for learning new techniques and new content; small modeling activities can be used to reinforce new concepts and to illustrate their applications; more extended modeling activities help students pull together ideas from different parts of a course and from different courses. Although time constraints still affect how often and for how long students can participate in the creative aspects of mathematical modeling, grade 9–12 students have a greater opportunity to pursue the full modeling cycle, including iterating to refine and extend the model. Improving upon one’s initial work is a defining practice in mathematical modeling, but it is a practice that may be quite new to mathematics students.

For most students who have not engaged in the kinds of modeling adventures described in this report, the emphasis on single right answers to questions, being taught a fixed body of specific skills, and reliance on mechanistic algorithms has given students an erroneous view of mathematics. A high school mathematics curriculum structured to contain separate courses in Algebra I, Geometry, Algebra II, and Precalculus contributes to a compartmentalized view of mathematics. Within such a structure, mathematical modeling broadens students’ views about mathematics and demonstrates vividly the connectedness

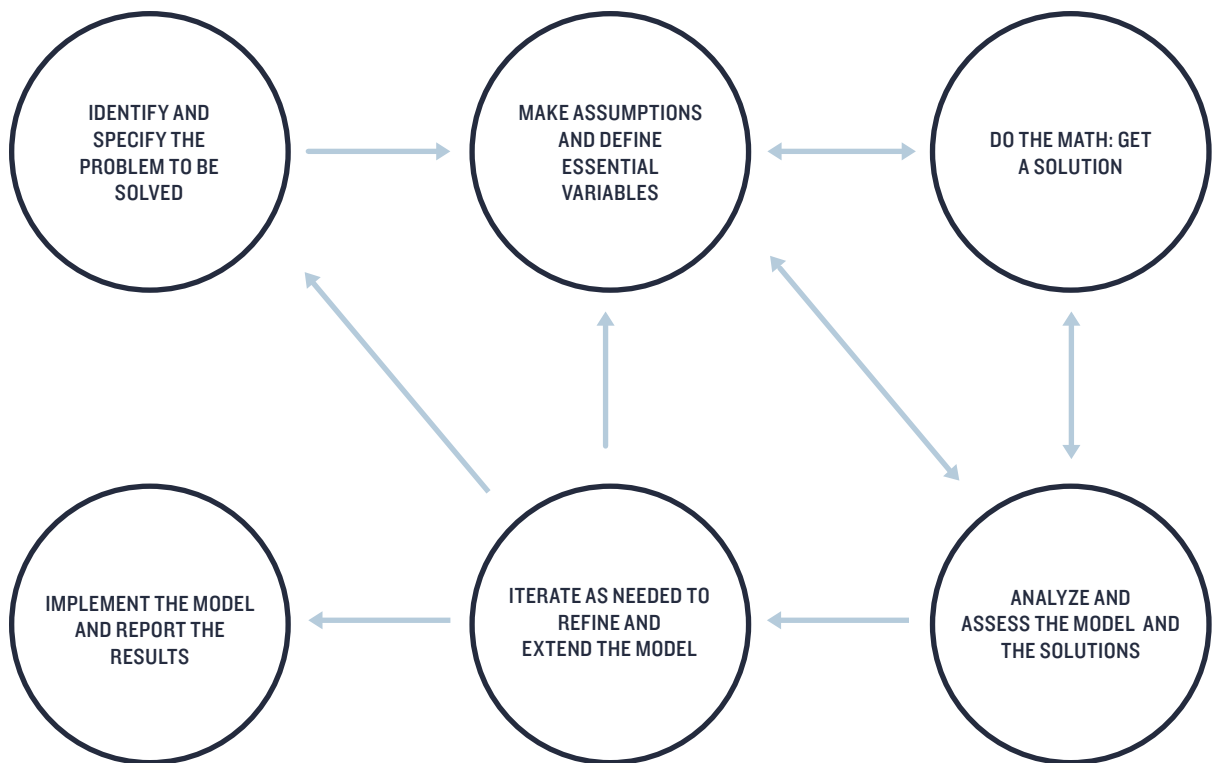


FIGURE 3.1: A SINGLE LOOP BACK AS ONE EXAMPLE OF A NON-LINEAR PASS THROUGH A MODELING PROCESS.

of the topics being studied, since models often involve a combination of geometric, algebraic, graphical, and statistical components.

As students begin to think about their plans for their futures and their individual interests in other fields are growing, the inherently interdisciplinary nature of modeling can help them see the broad applicability of mathematics. In modeling, students see how mathematics can help them make predictions and provide insight into the real world. Further, the real world can help them with mathematics; they can identify mathematical mistakes if their model does not produce results that are consistent with the real world. This back-and-forth helps elevate the status of mathematics; perhaps for the first time, students are able to see how math relates to the real world and can provide insight into many aspects of their lives. The power of mathematical modeling helps explain why, throughout human history, other fields and disciplines have drawn on mathematical ideas and methods. The important place mathematics has in the pre-college curriculum is a consequence of this power to explain and give insight into all areas of life. Whatever trajectory a student's life takes after high school, being able to think flexibly, to make realistic assumptions about what may happen when the future is uncertain, and to understand how conclusions are reached based on prior knowledge are very valuable life tools.

As they move through the high school curriculum, students become familiar with many of the standard types of mathematical models: linear equations to model constant change, exponential functions to model growth and decay processes, trigonometric functions to model periodic behavior, statistical models for data, and simple probability to model chance behavior. Using these tools, and especially when modifying them to match a given context, strengthens the student’s understanding of the mathematical ideas underlying these models and facilitates their use in contexts well beyond those in which they were first learned. In grades 9–12, the problems students investigate can be more complex, involve more variables, and offer a wider array of possible analytical and technology-based approaches than those explored in earlier grades. Students at this level may be more familiar with software and online applets, skillfully employing graphing utilities and spreadsheets, possibly writing short programs to approximate solutions where an algebraic solution method is unknown in the Do the Math: Get a Solution stage of the modeling cycle.

How modeling activities are organized in the upper grades depends on students’ prior experiences with modeling. If modeling, as described in the Elementary and Middle School section of this report, is a standard practice in the early and middle grades, classroom instruction in high school can take a more open form and build upon the understandings and skills developed earlier. If mathematical modeling is new to high schools students, much of the preparatory work described in the middle school grades will naturally fit into the student’s core curriculum. Good modeling scenarios can be used across grade levels and with both beginning and experienced modelers. Beginning students will need more support and encouragement, which often comes in the form of scaffolding added to the problem statements. However, too much scaffolding can reduce engaging modeling problems to a “paint by the numbers” experience, while too little scaffolding can result in excessive student frustration. Examples of effective scaffolding of students’ work are given in the exemplars in Appendix C.

**GUIDING
PRINCIPLES:
GRADES 9
THROUGH 12**

Chapter 2 presented five guiding principles for mathematical modeling in prekindergarten through grade 8. These guiding principles are:

- Modeling is open-ended and messy,
- When students are modeling, they must be making genuine choices,
- Modeling Problems can be developed from familiar tasks,
- Assessment should focus on the process and not on the product or pieces only, and
- Modeling happens in teams.

Each of these is equally important in the high school setting as in prekindergarten through 8, but they do not exhibit themselves in the same way in the high school classroom.

MODELING IS OPEN-ENDED AND MESSY

Modeling requires an open mind both from students and from their teachers and a willingness to explore, to fail and regroup, to revisit and improve, and to reflect on what their work says about the phenomenon under investigation as well as on what it does not say. This process is difficult for all students and can be particularly challenging to students with a history of success in the traditional mathematics classroom. Zawojewski, Lesh, and English, in describing the initial response of students to model-eliciting activities in engineering note:

“Frustration is often a first reaction on the part of students. For many, their past mathematics classroom experiences have led them to believe that when given a problem, they are supposed to be able to immediately search for, identify and apply the correct procedure. Thus, when they are unable to identify a particular procedure right away, they feel the problem is unfair, or that the teacher has poorly prepared the students for the task.”¹

Learning to support student work without being overly directive is a new challenge for many teachers and one that takes some time and experience to master. The essential understanding for teachers is that providing encouragement and support without providing “an answer” is essential for student growth in modeling. It is important for teachers to know that respecting student ideas and approaches, validating them when appropriate and supportively correcting when not, without being overly directive can be learned and can quite quickly become a normal pattern of interacting with students’ ideas. But it does initially take some willpower. Becoming comfortable with the level and kinds of support necessary can be difficult. Zawojewski, Lesh, and English continue:

“Students often ask teachers for help, especially during the first few model-eliciting activities. It is difficult for teachers to refuse to respond, yet the goal is to leave the problem solving to the students. Further, many students and teachers find it difficult to tolerate the inefficient approaches and wrong directions that typically surface early in the modeling episodes. A teacher may find that in the beginning, it is better to stay physically away from the groups, because students try to draw the teacher into telling them how to do the problem.”²

Throughout this messy process, students will devise approaches that will be new to the teacher, which can be both energizing and threatening to teachers new to the experience. The common saying in education, “you learn your subject most deeply by teaching it,” is made abundantly clear through mathematical modeling. Teaching students to model using mathematics is, in itself, a tremendous professional development opportunity for every teacher so engaged. As teachers of mathematical modeling in high school, it is amazing how much insight one gleans both from student errors and from student successes, when their

mathematical dreaming leads to approaches that are new and creative and enlightening. Modeling often reveals student thinking, including their misunderstandings, in ways that improve instruction. The experiences teachers have in working on extended modeling problems with their students increases the mathematical maturity and mathematical competence of everyone involved.

As one example of unexpected messiness, consider the *Free Throws*³ problem described below. It appears to be a standard homework problem in Algebra II, but not when the real world is taken seriously. The problem is presented as it actually happened, but the names can be updated to more current (or local) players if desired.

Michael Jordan is a basketball player from North Carolina. I was watching him play on television one day. As he drove for the basket, he was fouled. The announcer stated that "Michael Jordan is making 78% of his free throws." He missed the first shot and made the second. Later in the game, Michael Jordan was again fouled. This time, the announcer stated that "Michael Jordan is making 76% of his free throws." Determine the number of free throws Jordan had attempted and how many he had made at this point in the season.

PROBLEM 3.1: FREE THROWS

This problem was not intended to be a modeling session for the class, just a daily exercise. Working in groups, students took a few minutes to write out the equations for the appropriate success rates. Using x to represent the number of attempts and y the number of successful shots, for the initial success rate, they wrote:

$$\frac{y}{x} = 0.78$$

and to represent the rate after making one free throw and missing the other, they wrote:

$$\frac{y+1}{x+2} = 0.76$$

They dutifully solved to get $x = 26$ and $y = 20.28$. Thus, before that first foul, Michael Jordan had made 20.28 shots out of his first 26 attempts. You might be surprised at how many students were satisfied with this solution. Since their equations are correct and they performed the procedure they were taught correctly, students expect that their solution must also be correct. More thoughtful students recognized that x and y must be integers, and so they rounded the solution to 20 of 26 shots.

But not so fast! To four decimal places, the rational number $20/26$ is 0.7692, and 0.7692 rounds to 0.77, or 77%, not 78%. Did the statistician or announcer make a mistake? It is now time for a class discussion of the assumptions underlying our calculations.

If Michael Jordan had made 20 out of 26 shots, we would not expect the announcer to say, “Michael Jordan is making 76.92% of his free throws.” He would round the answer and say “Michael Jordan was making 77% of his free throws.” The messiness in the problem is that our initial numbers — 78% and 76% — are both rounded values. So, instead of starting with the equations

$$\frac{y}{x} = 0.78$$

$$\frac{y+1}{x+2} = 0.76$$

we should have started with the inequalities

$$0.775 \leq \frac{y}{x} < 0.785$$

$$0.755 \leq \frac{y+1}{x+2} < 0.765$$

Through whole class discussion, we arrive at the real mathematical model. Since the stated percentages were rounded, instead of a system of two linear equations, we have a system of four linear inequalities. Solving this new model involves a region bounded by four lines, a topic included in many Algebra II courses.

The problem changed in ways not anticipated by the teacher when he originally created the problem. The mathematics to be done had to be altered because, in modeling, we are required to “take the real world seriously.” This is a fundamental attribute of mathematical models.

WHEN STUDENTS ARE MODELING, THEY MUST BE MAKING GENUINE CHOICES

On a developmental level, high school students are actively seeking independence and validation of their individuality and ideas. Unlike traditional mathematics where there is a single path to a unique correct answer, the modeling experience — including the path, the solution, and the interpretation of that solution — are all shaped by genuine choices made by students. Therefore, incorporating mathematical modeling in your teaching gives you the power to invite your students to be active participants in creating mathematics of their own.

The extent to which the possible choices for high school students vary depends on the level of modeling being engaged in the lesson. As in the lower grades, sometimes students will be engaged in only a part of the modeling cycle, or a modeling context will be used to highlight a particular modeling technique that can be used more creatively in later experiences.

As an example of a problem using only a portion of the modeling cycle but allowing for a variety of student choices and decisions, consider the *Which Computer?*¹ problem.

The school board has decided that every mathematics classroom will be able to purchase a computer for demonstrations in the classroom. Your teacher has asked for your help to determine which computer to buy. The class finds a Consumer's Tips column that rates the different computers from which the teacher can pick. The consumer guide rates the computers from 0 to 10 on Performance and Ease of Use. A score of (0, 0) is terrible performance and very difficult to use while a score of (10, 10) is a perfect computer.

Which do you think is the best computer? Explain how you used mathematics to compare the computers.

COMPUTER	PERFORMANCE	EASE OF USE
A	6.4	8.5
B	7.3	7.5
C	9.3	3.8
D	8.8	6.0
E	7.3	6.0
F	5.5	9.7

PROBLEM 3.2: WHICH COMPUTER?

There are a number of ways of determining criteria for establishing which computer is the "best" computer. For all of them, a first step might be to graph the data. Looking at a long list of numbers rarely tells you what you need to know and often makes comparisons difficult. It is easier to see the data if they are plotted on a coordinate plane. The farther to the right a point is, the greater the performance of the computer it represents, while the higher on the plot a point is, the easier that computer is to use. Thus, we want computers that are far to the right and also high up. If a computer were both farthest to the right and highest up, we would certainly pick that one. However, higher performing computers appear to be more difficult to use.

From the plot it is clear that Computer C has the best performance, as rated by Consumer's Tips. It is also the most difficult to use. Computer F, on the other hand, is the easiest to use, but has the poorest performance. Computers A, B, D, and E all have good performance and are reasonably easy to use.

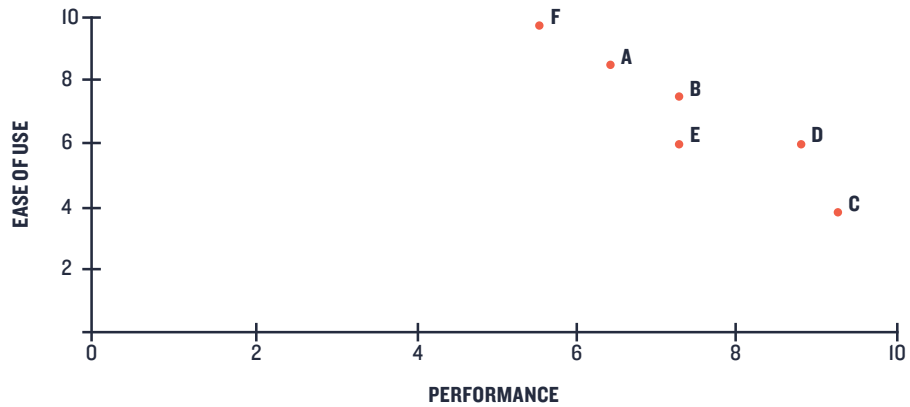


FIGURE 3.2: EASE OF USE PLOTTED AGAINST PERFORMANCE

Common student approaches involve finding the largest “total” measure. For some students, this means to find the largest sum of measures, while for others, it means the largest product of measures. In either case, students may calculate the resulting scores for the computers directly and obtain a final ranking for their criterion.

Using the sum of measures as the metric is equivalent to looking at graphs of equations of the form $P + E = k$, which are lines in the P - E plane. Using that interpretation, the best computer is the one on the line having the highest vertical intercept among lines with slope of -1. The “largest product” approach also leads to a different, but equally useful, geometric interpretation. In addition, students may invent still other criteria to define “best”, such as distance from (0, 0) or proximity to (10, 10). Students may wish to explore the connections between the symbolic and graphical interpretations of these other approaches as well.

Regardless of the approaches students choose, each method will allow them to make a recommendation to the principal. Their choices can be validated and their approaches compared graphically, and in the process, students pull together the algebraic statement of the comparison metric with its geometric representation.

MODELING PROBLEMS CAN BE DEVELOPED FROM FAMILIAR TASKS

Modeling offers a way to excite students about a variety of mathematical tools by using familiar contexts, both those that are “required” in the CCSSM and emerging tools not mentioned in the CCSSM that are finding use in new technologies and emerging applications. These include networks, difference equations, spreadsheet applications, probability arguments and statistical analyses, among many others.

As highlighted in the early grades, it is not necessary to do the whole modeling process in order to practice aspects of modeling. Snippets of modeling can be filtered into everyday lessons, when student are given 10-15 minutes to work together to extend the lesson and make some decisions about what to do and how to interpret their work.

In the *Which Computer?* problem, students had to decide how to put together two different measures and support their choice. The kind of information given in this problem, in which items are scored on several different dimensions, is very common and students will recognize this situation when comparing automobiles, televisions, and other consumer goods (including high schools and colleges).

In such scenarios, students may choose from a variety of ways to combine the two given measures for each computer, but these choices are limited. The focus of a lesson based on problems of this type may be on connections between the algebraic statements and the geometry in the plane represented by those statements. In such a setting, the modeling component is a means to an end, but the experience and many others like it help prepare students for modeling as the end itself.

By using bite-sized modeling decisions from familiar experiences like these, many of the important features of mathematical modeling can fit nicely into everyday lessons. Such modeling tasks help students make sense of the mathematics they are learning and vividly illustrate the importance of mathematics in understanding the world they experience each day.

ASSESSMENT SHOULD FOCUS ON THE PROCESS AND NOT ON THE PRODUCT ALONE

Lesh and Doerr, in describing the mathematical modeling done by engineering students on model-eliciting activities (commonly known as MEA's in engineering programs), note that:

“... these descriptions, explanations, and constructions are not simply processes that students use on the way to producing ‘the answer;’ and they are not simply postscripts that students give after ‘the answer’ has been produced. They ARE the most important components of the responses that are needed. So, the process is the product!”⁵

Although Lesh and Doerr were describing engineering applications of mathematical modeling, their observation transfers easily to all modeling experiences. The process is, indeed, the product.

This understanding of the essential nature of the modeling process is very important for teachers, students, parents, and administrators. All of these groups are familiar with the value of the products of mathematical work, where periodic timed tests are given to assess the student's competence on the techniques being studied. On such assessments, students expect to reach a proficiency level beyond, for example, 70% (whatever that might mean) every few weeks. Students are expected to be at least 70% proficient on the first three weeks of material, and equally proficient on the next three weeks material, and so on through the whole year.

But students are not going to be proficient in a complex, multidimensional, iterative task like mathematical modeling early in the game. They will struggle, and may feel like they are failing or will fail. Their parents may also feel their child is failing and believe that the teachers are not doing their jobs, since students are being assessed and graded on material that the teachers have, knowingly, not taught them.

The goals and processes of mathematical modeling need to be carefully and clearly explained to all the constituencies and the process of assessment and the role that assessment plays in evaluating the student's performance in the class (typically conveyed by a grade) must be made clear to all.

The student and parental concerns about grading can be lessened significantly if the modeling experiences are carefully scaffolded through the year – not just each individual assignment, but as a collection of assignments across the year. Initial problems start with more supports and give opportunities to build strengths early. As students progress, these supports are reduced to allow students to continue honing those skills while developing new ones. If the assessments include both feedback on the whole process and grades on the targeted skills, and early in the year, were scaled accordingly, then the goals of promoting the development of modeling skills and growing into the whole modeling process will produce much less push-back from students and parents driven by grades.

As described elsewhere in this document, mathematical modeling can be used to motivate new content, to apply previously learned techniques, and to pull together different mathematical concepts. In such lessons, the modeling is a means to some other end and is used to improve students' understanding of and facility with the mathematical tools taught in a specific course. It is difficult to assess the practice of mathematical modeling using a timed test given during a class period, but the command of content learned through modeling can be assessed using standard tools.

...one of the most important uses of modeling in the classroom is for the experience of creating a model all one's own.

While a modeling approach and modeling activities in the service of learning other content enriches the classroom experience, engages student interest, and is a good way to initiate the modeling process with students, one of the most important uses of modeling in the classroom is for the experience of creating a model all one's own. There is an excitement and a

level of ownership that can be transformational for students. As one high school teacher commented, "Math modeling doesn't just change their relationship with mathematics, but their view of themselves and their relationship with mathematics."

Assessing student comfort with and capabilities in the modeling process is a larger challenge than assessing students' knowledge of mathematics learned through modeling examples. The process of modeling, the interactions among the team members, the movement from one step in the process to another and back again all must be observed in real time as students work together on a challenging problem. "Assessment by walking around" is an important skill for the teacher to develop and is learned with time and experience. Listening in on student conversations, asking for a status report, and getting a history of the students' work—warts and all—allow the teacher to see the progress students are making in modeling, and also gives the teacher opportunities to support the students when they are stuck, building their confidence. For more insights on modeling assessment see Appendix D.

Good assessment problems require mathematics that is within the grasp of the students. The issue to be assessed isn't whether the students can solve the problem posed, but rather the manner in which they go about solving the problem. Remember, the process is the product.

The following examples help illustrate some possibilities.

This example is based on the *Driving for Gas*⁶ problem discussed in Chapter 1 as an example of a problem accessible to students early in their modeling experience. Here, in thinking about it as a potential assessment, we assume that students have not already worked on it.

Every driver recognizes the fluctuations in gas prices that happen almost on a weekly basis. Phone apps can map stations' prices and locations, and in some areas, a local radio station has a special report on the location of the gas station with the lowest price per gallon for regular gas. Of course, that station is likely to be across town from where you are driving.

If you know the locations and the prices at several gasoline stations, at which station should you buy your gas? Does it matter if you think that you are buying gallons of gas or that you are buying miles of travel? Develop a model that can be used by drivers of different cars that will tell them how far they should be willing to drive based on the specifications of their car.

Turn in four PowerPoint slides, one with the assumptions, one with the mathematical model, and two showing the screens for an app based on your model. The first app screen should request essential information from the user, and the second should show the app's response.

PROBLEM 3.3: DRIVING FOR GAS

Presenting and sharing their work in written and oral form is an important aspect of the modeling process and the feedback the students receive from the teacher and from their peers is important in their development as modelers. But writing a full report or paper on each project can be time consuming for students and can limit their opportunities to engage in modeling. The *Driving for Gas* assignment above shows a shortened reporting process for the project that is fun for students and reduces the time required for them in preparation as well as the time for the teacher in reviewing their work, but allows students to develop a full model.

Another alternative, again using the same context, shows how teachers might structure an assessment early in students' modeling experience by offering one or more specific examples for students to investigate. The process of creating specific examples is an important component of modeling with mathematics, and providing such examples helps students see their importance and utility. One example is shown below.

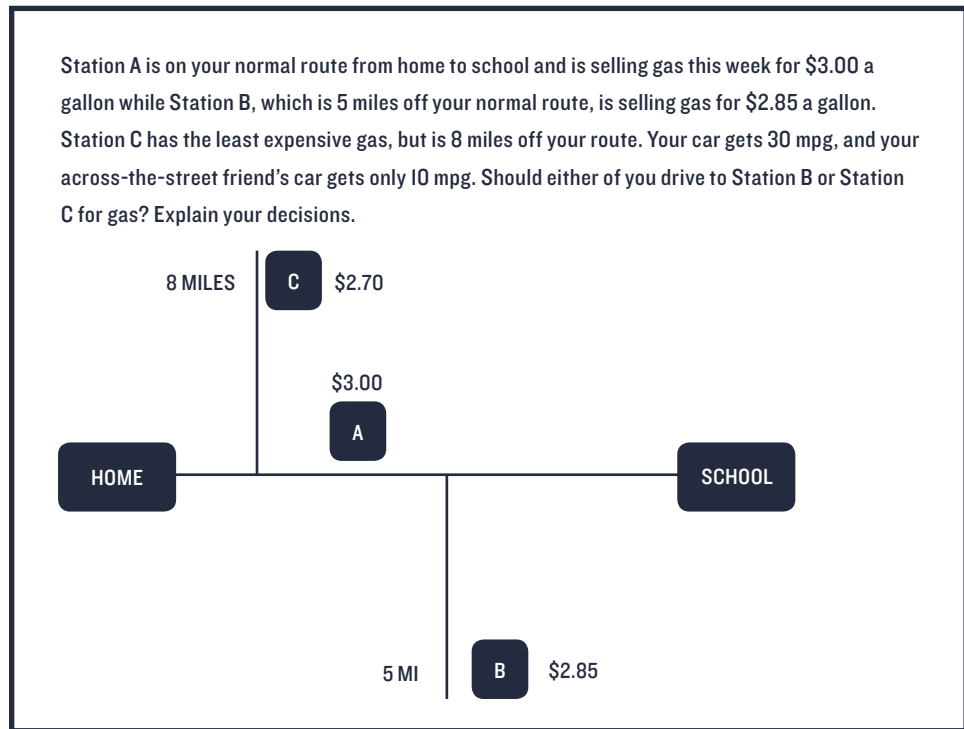


FIGURE 3.3: SPECIFIC 'DRIVING FOR GAS' EXAMPLE FOR INVESTIGATION

From this specific example, students can begin thinking about the important variables in the problem. Clearly distance and gas mileage play a role, but also hidden in the problem is the amount of gas to be purchased which is related to the size of the car's gas tank.

Although assessing the modeling process can be done by focusing on certain aspects of

the process in ways that fit within a standard class period, at some point each year it is important that students engage in the entire process over several days to experience putting all those pieces together.

Several approaches to solutions to the general *Driving for Gas* problem are presented in the exemplars section of the Appendix C of this document.

MODELING OFTEN HAPPENS IN TEAMS

In the US Department of Labor document, *Soft Skills Pay the Bills*, the ability to work in a team is listed third in the skills necessary in today's and tomorrow's job market.

“The ability to work as part of a team is one of the most important skills in today's job market. Employers are looking for workers who can contribute their own ideas, but also want people who can work with others to create and develop projects and plans.”⁷

Modeling problems encourage students to view standard techniques in new ways and to see how mathematics can bring insight and understanding to everyday situations. In working on a project, they also learn a lot about the topic of interest, such as ecology, or fairness, or engineering. In working in a group or team setting on modeling problems, students bring different perspectives, experiences, and skills to the process. They must merge these as they make decisions about which attributes are most important to model, how to represent those attributes (graphs, functions, simulations, etc.), and how to present their work to someone who has not been a part of the discussions. They pull together techniques from geometry and statistics and algebra and blend them in new ways, all of which makes those mathematical concepts and techniques more richly connected and understood and, with each use, more easily accessible in new contexts.

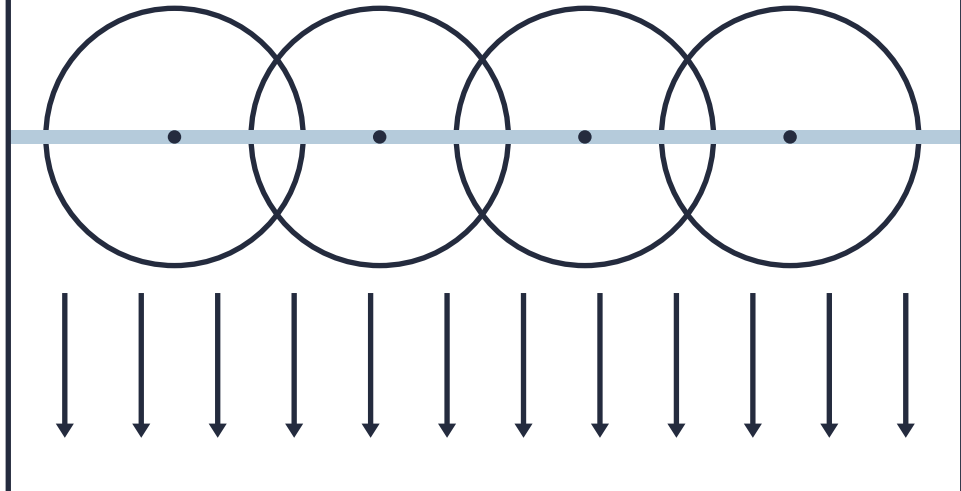
Modeling problems put a high priority on communication among team members as well as that between the team and those outside the group. Modeling activities support the English Language Arts Standards in significant ways and place an additional burden on English learners that must be attended to in the classroom. Due to the contextualized nature of mathematical modeling, these students may have a difficult time grasping the problem and communicating their findings. However, through the modeling process, these students also have an opportunity to further develop their English skills as long as you and/or their group members provide them with the necessary support.

The open-ended aspect of mathematical modeling, the variety of approaches to a problem, the need to present and interpret the ideas behind a model clearly, and the need to apply modeling in a variety of real-world situations all combine to create opportunities for individual students to contribute to a group's work in many ways. It is important to support students as they work together, to facilitate their cooperation, and to allow them to learn from each other.

The selection of teammates is an important responsibility of the teacher. Good modeling groups are diverse. It is often the collaboration of different skills, life experiences, and points of view that leads to the most interesting approaches to problems. Giving students opportunities to work in different groups and to play different roles in those groups is important in developing their modeling abilities. Thus all of the typical problems with student group dynamics must be considered when creating and supporting the student groups. As noted in *Using Work-Groups in Mathematics Instruction*, “Like any organizational plan, the value of work-groups lies not in the use of the method per se but in the quality of the implementation.”⁸ Modeling groups are no different, although usually the task presented for students is more engaging and motivating than standard classwork.

Teamwork is also important in many tasks by allowing work to be split up, making it possible to complete them within the allocated class time. *The Irrigation Problem*⁹ provides one example of how a group working together could develop a model that would be too long and tedious for a single student to complete.

A linear irrigation system consists of a long water pipe set on wheels that keep it above the level of the plants. Nozzles are placed along the pipe, and each nozzle sprays water in a circular region. The entire system moves slowly down the field at a constant speed, watering the plants beneath as it moves. You have 300 feet of pipe and 6 nozzles available. The nozzles deliver a relatively uniform spray to a circular region 50 feet in radius. How far apart should the nozzles be placed to produce the most uniform distribution of water on a rectangular field 300 feet wide?



PROBLEM 3.4: THE IRRIGATION PROBLEM

In this problem, students must realize that the amount of water sprayed on a particular point in the field by a single nozzle is proportional to the length of the chord of that nozzle's circle passing over that point. The areas of the field watered by two sprinkler heads receive an amount of water proportional to the sums of the chord lengths. Students can discretize the problem by looking at these chords every one foot along the pipe, and should recognize or be helped to recognize that they only need to consider the portion of the field between two adjacent nozzles.

Using facts from algebra or geometry, students will find that the length of a chord x feet from the center of a nozzle is $2y = 2\sqrt{50^2 - x^2}$. If nozzles are D feet apart, then the point that is x feet from one nozzle is also $D - x$ feet from the neighboring nozzle, allowing computation of the corresponding chord length in that watering circle.

Teamwork allows a group of four students to compute the lengths of all the chords at one foot intervals between two nozzles quickly. Students should realize that there is no distance between nozzles that gives a completely uniform distribution of water across the field. Thus they need to define some measure of uniformity of the distribution, perhaps motivated by examining the closest-together and farthest-apart extreme cases of nozzle placement.

**I'M NEW TO
MODELING. HOW
DO I START?**

As has been illustrated in the previous section, the modeling process is large, messy, time consuming, and can be both energizing and exhausting to students. It is equally, perhaps even more, large, messy, time consuming, and can be both energizing and exhausting for teachers. How do you make the transition from standard textbook problems to the full modeling cycle without becoming totally overwhelmed?

There are several guiding principles here as well:

- Start small
- Scaffold initial experiences with leading questions and class discussion
- Use common, everyday experiences to motivate the use of mathematics
- Use bite-sized modeling scenarios that require only one or two components of a full modeling cycle
- Share your goals and instructional practices with parents and administrators

START SMALL

Introduce modeling problems and activities gently. Begin with application problems and well-structured small investigations using leading questions to help direct student progress. Use the questions to give students some prescriptions for how to ask their own questions and how to modify their first ideas and improve them.

As one example, consider the problem of distributing disaster relief funds. For a more detailed discussion see Appendix C.

Hurricanes, earthquakes, floods, tornados, industrial accidents, plane crashes, and other similar disasters occur with distressing regularity and cause the loss of property, personal suffering and sometimes loss of lives. In the wake of a natural disaster, local or national governments (or private organizations) may set aside a dedicated “pot” of money (or clothing, tents, food, etc.) to help the people or communities affected by the storm with rebuilding or dealing with losses. Unfortunately, the size of the fund is often not large enough to cover all of the legitimate claims that are made against the sum of money allotted to try to relieve suffering.

Devise a fair system for the fund administrator to distribute a total amount of E dollars to the claimants.

PROBLEM 3.5: DISTRIBUTING DISASTER RELIEF FUNDS FAIRLY

For experienced modelers, the statement above might be all they need to get started. They would create their own examples and develop several procedures to compare. But for beginners, that would be totally overwhelming. So, we start small with some probing questions for students to consider. We make the problem similar to exercises students have done in the past, but with some open-ended questions at the end.

You might begin by giving the students a list of possible methods (Problem 3.6) and have them analyze and discuss them. This limits their ability to be creative, but it gives them suggestions for possible solution paths and helps them become comfortable with increasingly open-ended problems, opening the door to more creative modeling later. Don't be afraid to scaffold the early work in this manner, but don't continue the scaffolding forever. Scaffolding is an aid to modeling, particularly at the beginning, but the true modeling comes only after the scaffolding is removed.

After students have considered the definition of fairness underlying each model, ask them to create an example that might illustrate a weakness in each procedure.

How does each method handle more than two claims? Compare the methods and decide which you think is best. Suppose:

- \$1000 is available with claims of A: 200 B: 400 C: 600
- \$1000 is available with claims of A: 1000 B: 1200 C: 1600
- \$1000 is available with claims of A: 800 B: 1000 C: 1200

Finally, have the students create and defend their own method for “fairly” distributing the funds.

Suppose the pot of money is $E = \$210$ and Claimant A has verified claims of $\$160$ and Claimant B has verified claims of $\$100$.

Several methods have been proposed for distributing the needed funds. For each method, perform the indicated calculation and write two or three sentences to describe in everyday terms the definition of fairness being used.

- Entity equity: Each claimant should be treated as an entity, and the amount E is divided equally among all of the claimants.
- Equalize losses: This method gives each claimant an equal amount of loss. If we give A, a units, and B, b units and they have equal losses, we have the equations $a + b = 210$ and $160 - a = 100 - b$.
- Top off the requests: In this method, we consider the available funds as a fluid (blue in our diagrams) and the claims as glass containers whose heights are the size of the claims. We fill up the containers representing the claims with the fluid, with the filling stopping when one reaches the top of a claims container. The remainder of the fluid tops off the remaining claimants until all the funds have been distributed.
- Proportional Gains: Assign each claimant an amount proportional to the size of its claim. Since A's claim is for 160 and B's for 100, A's claim is 1.6 times that of B, so $1.6x + x = 210$.

PROBLEM 3.6: DISTRIBUTING DISASTER RELIEF FUNDS FAIRLY (CONTINUED)

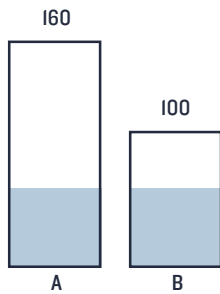


FIGURE 3.4: SMALL EQUAL AMOUNTS HAVE BEEN DISTRIBUTED TO THE TWO CLAIMANTS

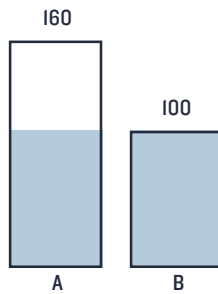


FIGURE 3.5: THE CLAIM OF 100 HAS BEEN MET

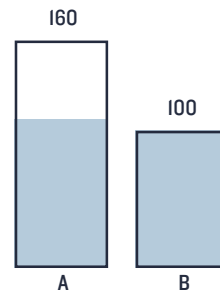


FIGURE 3.6: THE ESTATE HAS BEEN AS EQUALLY DISTRIBUTED AS POSSIBLE WITH A GETTING 110 UNITS

This problem, as presented here, has some characteristics of mathematical modeling and is something of a middle ground between an open-ended problem and a standard assignment. Students need to try several methods based on different definitions of and assumptions about fairness. Each approach is correct given those definitions. But the structuring of the

problem helps direct student thinking illustrating several ways in which the problem can be considered. Asking for other methods that were not a part of the problem is an important feature if the goal is to prepare students for future open-ended modeling activities. Small steps taken consistently can lead to large gains.

Appendix C contains an extended example of this problem in fair division, with a variety of methods ranging from those used in the Middle Ages to modern ideas worthy of the Nobel Prize. In addition to the models, the problem presents some of the rich history of the problem along with a bibliography.

SCAFFOLD INITIAL EXPERIENCES WITH LEADING QUESTIONS AND CLASS DISCUSSION

You might begin each problem with small group discussions to help students get started. After the initial discussion in groups, bring students together for an all class discussion so everyone has a clear understanding of the problem and some ideas for how it might be approached. Be sure to leave some approaches undiscussed. Then let students work for half a period and reconvene to see how they have progressed. Pulling the students together after some time working will help keep students moving forward, and will allow students to benefit from the issues other groups have encountered as well as from the advances they have made. Treating the student groups as separate research teams working in different directions but for the same goal and sharing information keeps the atmosphere collegial and supportive.

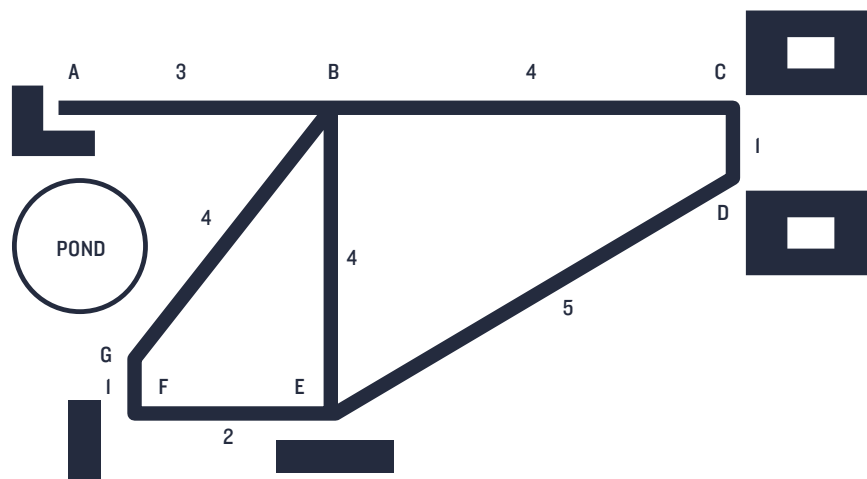
As one example, suppose you give *The Hot Dog Cart* (Problem 3.7) problem statement to an Algebra or Geometry class:

Have the student discuss the problem in small groups. You might seed the discussion with some specific questions to get started:

What is your definition of convenient and how can it be measured? What do you want to optimize when placing the stand? For example, do you want the shortest average distance for students? Do you want the shortest maximum distance to the cart? Pick some point on campus to be the origin and determine the coordinates of the different intersections. Can you find the point on campus that has the average x and average y coordinates. Does this location help in answering the question? What other criterion could be used?

Like the *Which Computer?* problem presented earlier, students must make some choices; in this case, about how to measure convenience. After having groups address these and similar questions, they should have several different ideas for how to proceed. Now, change the problem just a bit, and see if they can use one or two of the previous ideas to address the revised problem.

A map of a portion of a college campus is shown below. The map shows the walking paths and dormitories in this section of campus and the approximate distances (in 100 feet) between locations (the distance between the dorm at D and the dorm at E is 500 feet). Your roommate has convinced you to open a hot dog cart during lunch hours on weekends at one of the intersections along the walkways. You would like the location to be as convenient as possible for the students. Where on campus should you set up your stand?



PROBLEM 3.7: THE HOT DOG CART ¹⁰

The dormitories are located at positions A, C, D, E, and F and the numbers of students in each dorm are 200 in A, 300 in each of C and D, and 100 each in E and F. You would like the location to be as convenient as possible for the students. Where on campus should you set up your stand?

Other modifications: How would your choice for the location change if you knew that dorms A and C were female dorms and D, E, and F male dorms, and that only 30% of the females would likely eat at your stand while 80% of the males would likely eat at your cart? Or suppose the path between B and C and the path between E and D went uphill and that it is twice as hard to walk uphill as downhill.

Working through a series of questions like these helps students to see how different questions and different measures can lead to different solutions. By systematically increasing the complexity in the setting, students get comfortable with iterating their model and including the new, more challenging conditions.

USE COMMON, EVERYDAY QUESTIONS TO MOTIVATE THE USE OF MATHEMATICS

As noted earlier, small modeling problems can serve as motivation for learning new content and techniques. Probability, statistics, and working with data are playing a large role in modern life, and these topics are increasingly important in preparing students for work and college. Mathematical modeling and simulations can bring these subjects to life for students, and can lead to understanding without the formal approach that has often delayed the study of these topics. One small example can illustrate the power of simulation modeling and provide an introduction to a more formal analysis.

Sports are often a focal point of the high school experience. Whether students are on the teams or are even interested in athletic competitions, the performance (or lack of performance) of the high school teams is a common thread for students throughout the school, with a lot of excitement generated if the team makes the playoffs. Playoffs come in many forms. The Playoff Problem investigates the effect of the number of games on the probability that the weaker team will produce an upset.

The Super Bowl is a one game playoff for the championship of the National Football League. In baseball, the World Series is a best-of-7 game playoff, as is the National Basketball Association's Championship Series. The Woman's National Basketball Association uses a best-of-5 structure. In woman's tennis, the finals of the US Open is a best-of-3 series, while for men, it is a best-of-5 series. The winner in a series is the team that wins the majority of the games. Is an upset victory more likely with a short or long series?

PROBLEM 3.8: THE PLAYOFF PROBLEM ¹¹

At the introductory level, students can simulate the process of playing a best-of- n series using a random number generator on a calculator. In this setting, it is better to present a more specific problem, since we need numerical values in our simulation. For example, "Team A plays team B 10 times over the course of the season and wins 4 of the games. They meet again in the championship series. Does team A have a better chance of winning the championship if they play only one winner-take-all game for the championship, or if they have a best two out of three series?"

Since team A has won 40% of the previous games, we assume that the probability that team A wins a game is 0.4. With a graphing calculator, students can select a random number x , with $x \in (0, 1)$. If $x \leq 0.4$ then team A wins the game, while if $x > 0.4$ team B wins. This insures that team A will win approximately 40% of the individual games. By pooling their individual work, the students can "play" 100 single-game series and 100 best-of-3 game series and determine that team A has a slightly better chance of winning a one-game, winner take all

championship than a best-of-3. Similar simulations can demonstrate the fact that the longer the series, the less likely the weaker team will stage an upset.

Simulations are very good at illustrating the dynamics of a modeling situation and for developing the student's intuition about probabilistic situations. They do not allow the students to see why the results are what they are. For an understanding of the underlying principles being exhibited by the simulation, a more abstract model must be developed. The simulation is a good beginning to inspire the students to ask questions about why the simulations turned out the way they did.

USE BITE-SIZE MODELING SCENARIOS IN CLASS PRESENTATIONS TO ADD ONE COMPONENT OF THE MODELING CYCLE TO STANDARD PROBLEMS

Building up to a full modeling cycle takes time and experience. Students can develop that experience by consistently working on problems in class and for homework that involve small decisions and pieces of the modeling process as everyday activities. *The Mantid Problem* shown below appears to be a standard curve fitting problem, much like *The Free Throw Problem* appeared to be a standard system of linear equations. However, students must make a few important decisions about the how to handle the data and develop a function to represent the story being told by the data. Even so, there aren't too many different options available to them, so the problem is not wide open and fits nicely into a normal classroom lesson.

A mantid is a small, crawling insect that closely resembles a cockroach. Mantids are often used in biological studies because they move very slowly, so it is easy to keep track of them. Mantids move primarily to seek food. Researchers have been studying the relationship between the distance a mantid will move for food and the amount of food already in the mantid's stomach. The distance is measured in millimeters and the amount of food in centigrams (a hundredth of a gram). In the research, food was placed progressively nearer to a mantid and the distance at which the mantid first began to move towards the food was noted. The amount of food in the mantid's stomach was also measured. Measurements for 15 mantids are given below:

FOOD (CG)	11	18	23	31	35	40	46	53	59	66	70	72	75	86	90
DISTANCE (MM)	65	52	44	42	34	23	23	8	4	0	0	0	0	0	0

Based on the data provided, determine a functional relationship between the amount of food in the mantid's stomach and the distance it will walk to eat. Biologists call the amount of food in the stomach at which an animal will begin to seek food the hunger threshold. Based on your functions, what is the hunger threshold for the mantid?

PROBLEM 3.9: THE MANTID PROBLEM ¹²

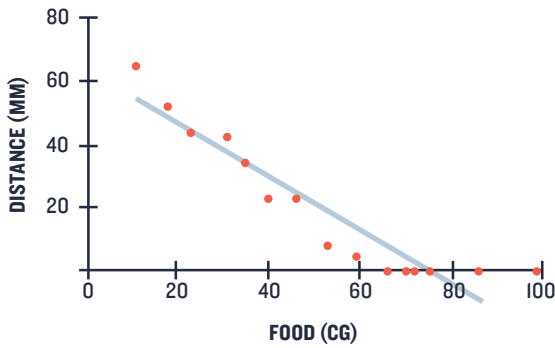


FIGURE 3.7: LINEAR MODEL

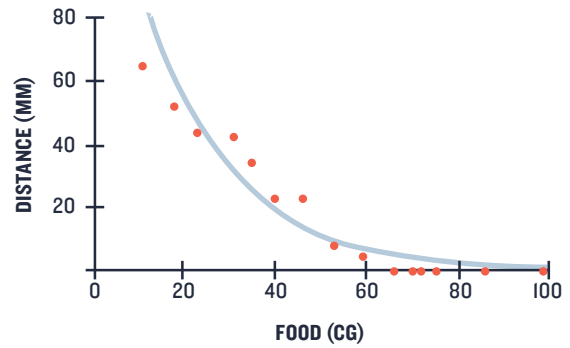


FIGURE 3.8: EXPONENTIAL MODEL

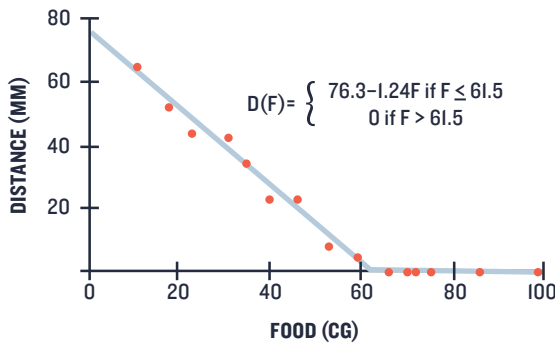


FIGURE 3.9: PIECEWISE MODEL

This problem requires students to make some decisions about how to use mathematics to describe a relationship between the two variables. One important decision is how to handle all of the zero distances shown in the table.

It is quite common for students to see the initial linearity of the data and fit a line to the entire data set, getting a linear model like that shown at the left. Others might consider the exponential model shown at right. Often, asking a few leading questions helps students see that the long string of zeros in the

data should not be included in the computation of the regression model since the model is intended to represent the relationship *when the mantid is willing to walk*.

Helping students form a refined problem statement in the *Identify and Specify the Problem to be Solved* component of modeling is important. As noted, the problem to be solved is, “how far will the mantid walk *when it walks for food*.” After all, we already know how far it walks when it doesn’t walk for food. This new problem statement could result in a domain limitation on the model or a piecewise-defined function as shown above. This part of the problem could be used either as an opportunity to recall prior work with piecewise functions or as an introduction to these important functions, since it illustrates nicely a situation in which they make sense.

In *The Mantid Problem*, although there are several approaches students can take to handling the zeros, there are not too many different directions in which they can go and there is no real follow-up other than a discussion of the different approaches and how students decided which approach was better. Small modeling activities like this one can fit nicely into the normal flow of the class and be used to help introduce new material, clarify some aspect that

might be confusing to students, or illustrate how new techniques can be applied or material taught in separate sections of the course can be combined. Many larger, more extensive modeling projects require combining several small models, so bite-size problems like this one can help students gain confidence and experience in their modeling decisions.

**DIFFICULTIES TO
ANTICIPATE AT
THE HIGH SCHOOL
LEVEL**

As was noted in the Elementary and Middle School sections, “Children often develop the misconceptions that a speedy solution is best and that every problem has exactly one correct answer.” The children are not alone in this misconception. Their parents often share the same beliefs about the ingredients of mathematical success. Consequently, students can find modeling frustrating long before they find it liberating. And the more advanced students can have the most difficulty, since they are often very well-prepared for the world of speedy solutions and one right path and have more years of success with the standard routine. Advanced students are often quite proficient at reproducing their teacher’s solutions and approaches quickly and with few errors. They may expect to know immediately what to do when faced with a problem and may not have developed strategies for figuring out what to do when they don’t know what to do. As noted earlier by Zawojewski, Lesh, and English, they may, in fact, feel that the task is unfair since the teacher never taught them how to solve the problem they were asked to solve. Importantly, their parents may feel the same way. The expectations in a classroom rich in mathematical modeling can be quite different from a classroom in which every problem has all the necessary information, a nice “clean” solution, and all numbers must be accurate to three decimal places.

Mathematical modeling offers a different view of mathematics and it opens doors to a different interpretation of what it means to be “good at mathematics”. Modeling can also provide a bigger range of students with opportunities to demonstrate competence and exhibit smartness in ways that are not always valued in math class (e.g., out of the box thinkers or students who are better at research and writing than they are at mathematics could make important contributions).

Just as the mathematical sophistication of the students increases as they move into high school, the level of mathematical expertise and experience of their teachers also increases in the upper grades. High school teachers of mathematics have spent a significant amount of time learning mathematics from mathematicians in their university studies. This extensive experience in university mathematics classes is a good thing, but it can create some substantial problems with respect to mathematical modeling. The instructional experience of teachers who major in mathematics often has a great imbalance in the attention to theoretical aspects of mathematics compared to attention to applications and modeling. Although this is slowly changing, mathematical modeling is currently absent from many undergraduate experiences and the applications most math majors see are almost exclusively limited to the physical sciences and engineering. As a consequence of

these experiences as students, teachers may see mathematics as a tool for the mathematical sciences but not necessarily for business, the biological sciences, political science, sociology or the other social sciences. Mathematics can be an important tool in the arts and humanities as well. Mathematics is truly for all. By focusing primarily on physics and

Our current approach to mathematics, with an emphasis on mechanics divorced from meaningful application, leaves many creative and mathematically capable students out.

engineering, we can present such a narrow view of the applicability of mathematics that it excludes the interests of many capable students who don't see themselves as future physicists or engineers, but who could profitably use their mathematical talents in a variety of other fields. Our current approach to mathematics, with an emphasis on mechanics divorced from meaningful application, leaves many creative and mathematically capable students out.

Mathematical modeling in the secondary schools can change this limited and limiting view of mathematics as a field by illustrating the power of mathematical models to gain insight into important areas of human need and by giving the students the energizing, even exhilarating effect of creative mathematical discovery. It is worth repeating: mathematics is for everyone.

Teachers whose education in mathematics was primarily abstract and theoretical may also have difficulties with the differences in the formal mathematics in which they were trained and its applications in modeling. In mathematics, if a theorem has several conditions and any one of them is not satisfied, then the theorem is of no use. In statistics, a subject that is at its core about modeling the world with mathematics, we have powerful theorems the conditions of which are never fully satisfied, yet we use them anyway: if the conditions of a theorem are nearly satisfied, then the conclusions are expected to be approximately realized. Statistician George Box wrote the oft quoted statement, "The most that can be expected from any model is that it can supply a useful approximation to reality: All models are wrong; some models are useful" in his seminal text on the design of industrial experiments, *Statistics for Experimenters*.¹³ Nonetheless, as students move into post-secondary education from high school, they will acquire new mathematical tools which will enable them to get somewhat more realistic approximations of the world around them — as messy as that world and its problems may be.

4

MATHEMATICAL MODELING AT THE UNDERGRADUATE LEVEL

INTRODUCTION

In the previous chapters we discussed how mathematical modeling could be infused throughout the prekindergarten through grade 12 curriculum. But the setting of undergraduate mathematics allows for the introduction of complete modeling courses as well. We feel strongly that one such course should be available to entry-level students and so we begin this section by discussing features of a course in mathematical modeling that does not require prerequisite mathematics beyond the high school curriculum. We also argue for and discuss how to weave open-ended modeling problems into the STEM-major curriculum beyond the introductory level.

We sincerely hope and believe that mathematical modeling will gain a more prominent place in US mathematics education at all levels. And so, we feel that this chapter will be useful not only to undergraduate faculty in mathematics, but to teacher educators in general, whether teaching math content courses or not.

We conclude by revisiting our Guiding Principles to see how these relate to our teaching modeling at the undergraduate level.

AN INTRODUCTORY COURSE IN MATHEMATICAL MODELING MOTIVATION

Consider the audience of students who will only take one mathematics course in their undergraduate career, and imagine you have been asked to develop that course. What should you include? Before answering this question, consider the 3-year question and the 20-year question:

THE 3-YEAR QUESTION

What skills may be important for these students during the remainder of their college career?

THE 20-YEAR QUESTION

What do you hope that your students remember from your class 20 years after they've taken it?

In the context of these two questions, providing students with an opportunity to develop not only an appreciation for the applicability and utility of the field but also a set of transferable skills has the potential to be far more impactful on their futures than teaching them a specific set of mathematical skills, such as college algebra, in isolation from application. While specific mathematical skills from a college course may be easily forgotten, the powerful experience of tackling a real world problem can help students develop a lasting tenacity and confidence such that they are better equipped to address the ill-defined challenges found outside of the classroom. Through the process of researching topics, comparing the reliability of their data sources, and making and justifying their assumptions, students are forced to make choices that matter as they work their way towards one of many possible solutions, and this experience is particularly empowering for students in contrast to a typical mathematics course in which there is one right way to reach a uniquely correct answer. Additionally, after completing such a course, students will have had a chance to see the broad applicability and value of mathematics through first hand experience. Even if they, themselves, never choose to use mathematics to solve a problem after the course, they will be more likely to hire a mathematician to solve a problem they encounter in their business, vote for funding for the mathematical sciences as an elected official, and encourage their own children to enjoy the study of mathematics.

The focus of the applications can be selected to harness the interest and enthusiasm of your students.

The focus of the applications can be selected to harness the interest and enthusiasm of your students: the interdisciplinary topics might be selected based on popular majors at your institution, regional concerns, or global

challenges. Students can solve the problems they encounter in a modeling course using the mathematical skills they already have or they can develop new ones in the context of the problems; therefore, no prerequisites need be imposed on such a course.

In a mathematical modeling course, the focus is broader than just the mathematics or even the modeling process; the focus grows to include transferable skills. In developing a course, you can decide goals for the level of exposure and mastery for each of the following transferable skills.

DISTILLING A LARGE, ILL-DEFINED PROBLEM INTO A TRACTABLE QUESTION

Most real-world questions of consequence are complicated; perhaps it's not even clear what question is really being asked. It can initially be intimidating for students to investigate such questions. Thinking about which quantities matter the most and how they might be

related is a valuable skill that students can use throughout their lives. Students should be encouraged to make defensible assumptions in order to reduce the larger problem into one they can solve. In solving this smaller problem, they may be able to make insights that help them address the original problem. Targeting this skill prepares your students to apply the modeling process outside your course, where problems are not presented in a manner that would necessarily point towards a mathematical approach.

Students solving the Gas Problem in Chapter 1 had to form a tractable question in order to make progress answering the question. Another example demonstrating this is the Elevator Problem (Problem C.1), which appears in Appendix C. This question asks students to determine how one should program elevators so workers get to work efficiently.

IDENTIFYING AND USING RELIABLE RESOURCES

Many questions require students to do some research to understand the problem and to acquire datasets to use in their models. It is important that students acquire the ability to search the internet and the library to find sources and to measure the relative reliability of sources. Sometimes real data is incomplete, or the most complete dataset is known to be biased in a particular direction; students can use sensitivity analysis to explore how using biased data could impact their results. Targeting this skill prepares students to be more independent problem solvers, as they are able to find resources and learn on their own; it also gives them confidence in using real data, even when they know it may be flawed.

WORKING EFFECTIVELY IN A GROUP

Truly challenging problems are best addressed through collaboration, as students each bring their own ideas to the modeling process. Therefore, a course in modeling can be the perfect place for students to practice effectively managing the dynamics of working in a group, which includes utilizing the strengths of each team member, as well as exploring ways to leverage interpersonal relations to maximize the productivity of the group. While group work can present challenges, overcoming these challenges can become an overtly stated goal of the course, instead of a distraction. Targeting this skill prepares students for working effectively with others in the workplace and perhaps in other college courses.

COMMUNICATING EFFECTIVELY

An important step of the modeling process is communicating one's findings to others. As students seek to communicate their ideas, it is beneficial for them to consider the goals of the report, the level of detail required, the audience, the level of formality, and the medium. Depending on the goals of your course and the framing of the problem, the communication could be

- unbiased or persuasive;
- a presentation of the results or a discussion of the process;
- geared towards a technical audience or a general audience;
- formal or informal;
- oral, written, visual, or mixed media.

In all of these cases, asking students to consider the audience that would be interested in knowing the answer to their question makes the work of finding the answer more palatable, and reminds students that finding an answer is not always the end of the problem. They also need to be able to convey their ideas and solutions effectively. The development of communication skills transfers directly to other courses and the career marketplace.

These broadly applicable skills are precisely the types of skills employers look for in job-seekers. Further, they are life-skills individuals can use to help them better navigate the global community in which we live, which will continue to present generations of students to come with challenges we have yet to imagine.

IMPLEMENTATION

As with the development of any course, it is important to consider the goals of the course. Typically, as mathematics faculty, the goals for our courses are a list of mathematical concepts to understand and mathematical skills to master; often these topics are aligned with an established textbook in the field. When designing a course in mathematical modeling, you may still have some specific mathematical goals in mind, but the larger focus is on the modeling process itself, as well as the aforementioned development of transferable skills and habits of mind. Additionally, many textbooks on mathematical modeling are more closely focused on specific mathematical models (as nouns) rather than on the process of modeling (as a verb). Therefore, the design and implementation of a modeling course may feel very different from the design and implementation of other mathematics courses. In this section, we explore issues related to the design and implementation of a class focused on mathematical modeling, specifically the organization of the content, the ways in which students may engage with the problems, and the ways in which you, as the instructor, can facilitate student development and learning.

Appendix C includes a variety of modeling problems each with a thorough discussion of how students might engage with those problems, how faculty can help facilitate the student modeling process, and ways the problem can be modified to achieve specific learning objectives.

ORGANIZING CONTENT

Model Type While there are many textbooks whose titles address mathematical modeling, many of these are organized by model type (linear, exponential, etc.), and this approach presents potential problems for a course on the modeling process. To highlight these problems, let's briefly explore an example. Suppose you are teaching a course organized in this fashion, and the class is learning about linear models. You present an open-ended question, but your students feel obligated to use linear models to solve the problem. While this isn't necessarily bad, it can lead to poor modeling decisions. For example, as your students search for data, they may come across data that is nonlinear and discard it; even

worse, they may force this nonlinear data into their linear model. So, while it may seem natural to us, as mathematicians, to organize a modeling course by looking at separate types of models, this approach can be dangerous if poorly implemented. Therefore, we offer two alternative course organizations below followed by some ideas to keep in mind as you develop your course.

The Modeling Process In a course where the main focus is on the modeling process, it is natural to organize the course by breaking down the process such that students can practice each element individually before ultimately using all of their new skills to solve an entire problem. Recall that the modeling process is not a set of numbered steps, as there are problems for which the process may require additional iteration or an unusual ordering of the components. Therefore, as you organize the course, you can have students engage with these components in a way that you believe best supports their mastery of the entire process. Below are three examples of implementation ideas for such a course.

- Early in the course, students could read problem statements and practice the early components of the modeling process by brainstorming a list of important factors, searching for data sources related to their list of factors, and refining their list of factors based on their research, all while keeping track of their assumptions. After students have sufficient opportunities to practice this and feel comfortable with these phases of the modeling process, they can begin to move into a phase of developing and using mathematical models.
- Early in the course, faculty could present their students with the rubric against which student writing will be assessed. Students could then read and assess (grade) executive summaries, perhaps ones from previous students or ones found online from published government reports, before ever attempting to write their own.
- Students could spend a portion of the course examining existing mathematical models and their inputs to recover the assumptions that went into those models. Later in the course, they could then use what they learned previously and apply it to new problems, constructing and justifying their own assumptions as they develop their models.
- You can start the course by presenting students with problems that are less open-ended and work your way towards more open-ended problems.

Thematic Approach Another way to organize the course is to have students repeatedly engage in the entire modeling process while tackling increasingly more complex challenges focused around a central theme such as social justice or sustainability. This topic should ideally be selected such that relevant problems can be pulled from a variety of disciplines, highlighting the broad utility of mathematical modeling. The course could further be divided into units by exploring different subfields of the central theme; for example, if exploring sustainability, perhaps a portion of the course could focus on ecological sustainability, another on energy sources, and another on international policy. In this way, students could see that modeling can be a powerful tool in helping to answer questions across a variety of subfields within a discipline.

Other Considerations Regardless of how you decide to organize the course, there are some models that are so ubiquitous, such as linear models, it may behoove your students to get exposure to some of these during the course; a few of these are listed below.

Recall that we presented challenges to courses based on model-types, such as students forcing nonlinear data into a linear model simply because they are in a unit on linear models. However, this pitfall doesn't mean you should avoid linear models in your course! Instead, you can give students a problem that will most likely lead them to a linear model. Once they reach their results, you can have the class reflect on the properties of the linear model they created, such as how changing the rate (or slope) influences the results or the physical interpretation of changing the intercept.

The most common models and modeling tools include:

- Linear models;
- Nonlinear models, particularly exponential, logarithmic, and polynomial models;
- Discrete dynamical systems for modeling change;
- Optimization;
- Statistics.

STUDENT GROWTH

A student progresses in her ability to model, not only through practicing the modeling process, but also by increasing the arsenal of mathematics that she can apply. For example, many modeling scenarios involve:

- Change
- Chance
- Finding optimal decisions
- Finding trends in data sets
- Making decisions involving chance and risk
- Developing strategies for decisions where the outcome and payoffs to the decision maker depend not only on his decisions but the decisions of one or more other players.

A modeling experience can be designed to provide both practice in modeling and growth in the mathematics necessary to model and analyze these situations. Any of the following subjects can be presented to students with only a high school preparation in mathematics. A modeling project can easily be included for any of the topics.

Modeling change with discrete systems

Content: Students use the simple construct “the future equals the present plus change” by making assumptions on the change during an appropriate discrete time period. The students can then build numerical solutions of difference equations and systems of difference

equations using spreadsheets. They can examine implications of their model for stability and long-term growth. Analytical solutions can optionally be studied.

Example: You plan to invest monthly to pay for your children's educations. You want to have enough to withdraw \$2000 each month for 8 years beginning 15 years from now. Build a model to determine how much must be invested at different interest rates assuming you stop investing when the first child enters college.

Proportionality and geometric similarity

Content: Students use the definitions of proportionality and geometric similarity to construct explanatory models of interesting behaviors. With a spreadsheet or graphing calculator, students can examine behaviors such as a “rule of thumb” that appears in a driver education text that recommends 2 seconds between automobiles at all speeds. Does that make sense? Is it safe at all legal speeds?

Example: The Navy's weight standards allow a constant 5 pounds for each additional inch of height. Assume two muscular sailors are geometrically similar. But one is short and the other is tall. Which will have the most difficulty making the standard? Build a model that *explains* weight as a function of height for geometrically similar individuals. Find data to test your explanatory model.

Model fitting

Content: Students now have models that they have formulated to explain a behavior. How do they fit the model to a set of data? Students learn different criteria of “best fit” to determine the parameters of their proposed models.

Example: A student has reasoned that weight should vary in a cubic fashion for geometrically similar individuals. Given a set of data, what cubic best fits the data? Is there a pattern to the errors that results from your fit?

Experimental (empirical) modeling

Content: Often behavior is too complex to explain mathematically, but an examination of the data reveals a pattern. Students learn how to build a difference table to gain insight into the nature of the data. If appropriate, they can smooth the data with low order polynomials, splines or other functions. They examine the residual errors from their empirical model to refine the model if a trend is evident in the residual errors. Often, they gain insight into the underlying cause of the behavior.

Example: Given the data representing the weight-lifting results from the last Olympics, build an *empirical* model that *captures the trend* of the data to predict total weight lifted as a function of the contestant's weight. Use your model to determine pound for pound, who was the strongest lifter. Now use *geometric similarity* to explain the behavior.

Modeling with simulations

Content: Students study random number generation and Monte Carlo simulation. They

apply the ideas first to deterministic behavior (such as the area under a curve) and then to probabilistic behavior (inventory and queueing problems).

Example: Construct a Monte Carlo simulation of the the card game Blackjack (twenty-one). Develop and test a strategy for playing the game. What is the expected value of your strategy?

Discrete probabilistic modeling

Content: Students learn to analyze behavior which varies in a probabilistic manner. They can build models of systems to determine the reliability of the system.

Example: Lumbercutters wish to use readily available measurements to estimate number of board feet of lumber in a tree. Assume they measure the diameter of the tree at waist height. Develop and test a model that predicts board feet as a function of diameter in inches using regression models.

Optimization of discrete models

Content: Many scenarios require a “best” solution given an objective and constraints. Geometrical linear programming can readily be taught. Students can build models to approximate a behavior with a linear objective function and linear constraints.

Example: Build a model of an investment portfolio that maximizes yearly return on a specified amount while keeping a minimum amount in stocks, a liquid savings between specified limits, and the amount in stocks not exceeding the combined total in bonds and savings.

Modeling using graph theory

Content: A graph can be a powerful tool for determining interesting behaviors such as connectivity.

Examples: (1) Determine the Bacon number of an actor such as Elvis Presley, Babe Ruth, etc. (2) The manager of a recreational softball team has 15 players on her roster. A starting line-up requires 11 players at specified positions. Given the positions that each of the 15 players can play, is it possible to have a person at each position? If so, how many ways can the manager assign players to each position?

Modeling with decision theory

Content: Often the outcome of a decision is determined partly by chance or nature. In choosing among alternatives, what criterion is appropriate? How do we measure the risk inherent in a decision? Students learn criteria such as expected value, minimax, maximin, maximax, etc., that are appropriate in different situations. They learn to build decision trees to model a decision.

Examples: (1) Should you join a social network such as Facebook? Build a decision tree to model your decision. (2) Should U.S. citizens build their own retirement through 401 K's or use the current Social Security program?

Modeling with game theory

Content: In game theory, the outcomes and payoffs depend not only on the choice of the decision maker, but also on the choice of one or more other players. Students can learn games of *total* and *partial conflict*. If communication is possible in the scenario, strategic moves including commitments to first moves, threats and promises are options the decision maker must consider.

Example: Research the Cuban Missile Crisis of 1962. Determine the possible strategies of the United States and the Soviet Union. Assign ordinal values to the outcomes. What would be the result if each side played it maximin (conservative) strategy? What strategic moves (commitments, threat, promises, and combinations) are available to each player? Now analyze the actual sequence of decisions by each country.

Families of functions and their graphs

Content: Students learn to use the graphs of function as models to gain qualitative insight into a behavior.

Example: Build a graphical model to analyze the following quotation: “The effect of a tax on a commodity might seem at first sight to be an advance in price by the consumer. But an advance in price will diminish the demand, and a reduced demand will send the price down again. It is not certain, therefore, after all, that the tax will really raise the price.”¹

WHAT DO STUDENTS DO IN A MATHEMATICAL MODELING COURSE?

Unlike a course on traditional topics in which students may listen to lectures or follow examples to solve a set of problems, students in a mathematical modeling course tackle big, messy, open-ended problems without any textbook examples. This means students must be much more personally engaged. As students work through the modeling process, they start with an open-ended question. They identify the most important factors and develop reasonable assumptions about the relationship between those factors. They find sources of data and preprocess the data into a useful form. They run their models, examine their results, draw and justify inferences and conclusions, and then revisit their assumptions, perform sensitivity analysis, and refine their model as they try to address the original problem to the best of their ability. They synthesize their findings and report their results appropriately – aimed at the appropriate audience, with the appropriate tone and formality, and using the appropriate medium and format. Without a road map or an example from a text or their notes, students are faced with the challenge of decision making at every stage in the process.

While students should take ownership of the decisions throughout the modeling process, they need not do so in isolation. As previously mentioned, truly important modeling problems are best addressed by groups of students working closely together. By working in teams, students get an opportunity to have their individual ideas validated or respectfully challenged by their teammates; they also get the opportunity to participate in the thoughtful

critique of ideas from others in the group. For certain problems, particularly those for which you expect a diverse set of approaches, it may be helpful for groups to work for a while on their own and then share their ideas with the whole class, as this allows groups, who may have become focused on a single approach, to pull back and revisit the problem from a broader point of view. This class-wide sharing approach can be used: early in the modeling process, as students identify the key factors and assumptions; during the middle of the process, once students have developed a mathematical model; and at the end of the process, as students are making sense of their results in the context of the final problem. Furthermore, these class-wide idea-sharing sessions can provide an opportunity for students to practice their ability to communicate their ideas in an informal setting.

This idea of sharing also needn't be restricted to just your classroom. In the spirit of interdisciplinary study, students can look for local data sources by consulting with other faculty or representatives from nearby businesses; they can also leverage the library and the internet to obtain data sources from reputable sources like national agencies (NASA, USGS, USDA).

Giving students the freedom to define problems in their own way can lead to genuinely new and creative models...

We cannot stress enough the importance of having your students be the ones who shape the messy problems into tractable ones, figure out what data they need, and then go out on their own to find it. Giving them the freedom to define problems in their own way can lead to genuinely new and creative models, unlike a traditional course where students are simply working out variations of the same problems that have been solved by generations gone before. This push towards self-sufficiency may cause students to feel initially uncomfortable and even rebellious, but as students start to gain traction, the sense of accomplishment outweighs the struggle, and students become grateful for the experience.

FACULTY ROLES

Now that we've presented some ideas of what your students would actually do in the course, it is important to address how you, as the course developer and instructor can support your students in their work and in becoming strong modelers. While there are many ways in which you could provide support, below, we specifically address the following roles of a faculty member involved with such a course: problem developer; course developer; classroom facilitator; coach.

PROBLEM DEVELOPER

We have the opportunity to present students with questions that they may never have thought they could use math to answer. Early in the course, it is important for you to present challenges that find a balance between being approachable enough that your students can gain traction while being daunting enough for your students to feel a sense of

accomplishment when they gain traction. As you develop each problem, you should consider the following questions:

- How approachable but daunting is this problem for my students?
- How open-ended versus guided is this problem? How closely does this problem mimic the ill-defined problems with unclear objectives that my students are likely to face in life?
- How easy will it be for my students to find data for this problem? Reliable data? Complete data? Formatted data?
- Are there one or two relatively obvious mathematical approaches to this problem, or is there a wide variety of mathematical approaches that are equally likely to be taken by my students? (Appendix C gives good examples of the latter.)
- What technology might my students need to address this problem, are they prepared to use that technology, and do they have access to that technology?
- What are the format, formality level, audience, and tone that my students should use in reporting their results for this problem?

Note that there is no “right” answer to the above questions. Rather, the appropriate answer will vary based on the goals of your course, how far you are into the course, and the personality of the particular students in your class.

If you find that your students are not engaging with the problem as you’d hoped, a problem can be easily adjusted, even as students are working on it. For example, if students are really struggling, you can offer nudging guidance; if students are not challenged by the problem as stated, you could ask them to argue an alternative point of view or apply their findings to another problem. Also, while the ultimate goal is for students to develop the skills and confidence to tackle a very open-ended problem, you may decide that your students initially require a bit more guidance. As instructors, we may instinctively try to anticipate all of our students’ needs and include all of the information they will need. However, this decreases their cognitive load and robs them of the opportunity to overcome messy challenges.

In terms of technology, real world data rarely lends itself to “nice” solutions; therefore, exposing students to easily accessible software, such as Microsoft Excel® or freeware equivalents, can allow students to address challenges that would be unsolvable without technology. Depending on the goals and structure of your course, it may be worth the time investment to bring students up to speed on more advanced technology options such as mathematical packages, statistical packages, or programming languages; it is important to recognize that the instructor will need to invest time in developing their abilities with these tools, and this time comes at the cost of other course material. Again, there is no right answer for this balance; we just encourage you to make the decision with purpose and awareness of the issues.

As you develop problems, you might want to draw on local resources, such as other faculty, campus services (dining, recycling, facilities, etc.), or local businesses, for problem ideas. If you are not tied to local problems, there are several existing resources for modeling problems; problem development is a lot of work, and there is no need to reinvent the wheel when there are many great problems to get you and your class started. Given the ever-changing landscape of the internet, some of these resources may not be available for the indefinite future, but an internet search may be rewarded with additional resources.

- COMAP (www.comap.com): The Consortium for Mathematics and Its Applications is a non-profit organization that has several problem banks on its website:
 - › HiMCM Competition Questions: COMAP hosts an annual 36-hour high school mathematical modeling competition. Although these questions are aimed at a high school audience, they may be modified to fit your college course. All the questions from prior years are posted. These are freely available.
 - › MCM/ICM Competition Questions: COMAP also hosts an annual 96-hour undergraduate competition in mathematical modeling (MCM) and interdisciplinary modeling (ICM). All the questions from prior years are posted; these are freely available. Judging commentaries and winning papers are available with a subscription to COMAP.
 - › Undergraduate Mathematics and its Applications (UMAP) modules: These are mathematical teaching modules that include applications that could be modified to become problems for your class. These are available with a subscription to COMAP.
 - › Interdisciplinary Lively Applications Projects (ILAPS) projects: These are interdisciplinary projects that include full commentary for instructors. These are available with a subscription to COMAP.
- SIAM/M³ Challenge (m3challenge.siam.org):
 - › M³ Challenge Problems: SIAM runs a 14-hour high school competition in mathematical modeling. Although this competition is aimed at the high school level, the questions may be modified to meet the needs of your college course. All prior questions are freely available. Additionally the site offers free access to examples of winning students submissions.
 - › M³ Sample Problems and Thought of the Month (TOM): To help teams prepare for the challenge, the M³ problem development team also offers sample problems with insights for both students and faculty, as well as TOMs, which are problems where there are opportunities for interested students and faculty to discuss a problem with the developer of the problem in an online forum. All of these resources are freely available.
 - › *Math Modeling: Getting Started & Getting Solutions*: While this modeling handbook was not designed to serve as a source for modeling ideas, it is an affordable (free online access) and detailed explanation of the modeling process that could be of value to both you and your students.

COURSE DEVELOPER

As you develop your course, you should consider which transferable skills and mathematical models you wish to address. The organization of the content and the types of reports you require should largely be shaped by the primary goals you have for the students in your course. It may help to actually write down your answers to the 3-year question and the 20-year questions, and use these answers to guide the overall course development. You will also want to consider the pace of your course. Since the modeling process is elaborate, it takes practice to judge the status of works in progress. Therefore, during the first iteration of teaching a course, it may be better to have students engage with the problems at what may feel like a slow pace, until you are more comfortable at gauging their progress.

While there are clearly challenges in designing a mathematical modeling course, it is also a rewarding experience. There is so much flexibility in selection of interdisciplinary and disciplinary topics, especially if the course is a terminal course (i.e. there is not a required course that follows your course) and if there are no prerequisites. As the course developer, you have the rare combined opportunities of bringing your own interests and passions to the classroom and leveraging the interests and passions of your students, as well as leverage the special areas of expertise of your campus. For example, if your school has a strong marine science program or a world-renowned Egyptology lab, you might want to work with faculty in these areas to develop modeling problems around these themes.

CLASSROOM FACILITATOR

In the classroom, as students work in groups to address the challenging problems you've developed and presented, you will often step into the role of facilitator. This role involves walking around the room, listening into group discussions, and assessing the levels of productivity for each group. When you overhear one or more groups heading down a path of poor decisions, it can be tempting to intervene directly and show them the error of their decisions and point out better options, but this stifles creativity, not only for that problem, but also for future problems by squelching student confidence. There are many alternative solutions, and as you decide which to choose, you should consider the current tenacity level of the struggling group(s) and decide on the critical level of support or guidance needed to nudge them to reconsider their missteps. One approach is to take a Socratic approach with the group itself, asking them to justify their work to you or asking them to consider an example at an extreme that might highlight the errors in their current thinking. Often when the group shares their ill-conceived ideas aloud, they are able to identify the issues and address them on their own. Another approach is to bring the class into a whole-class discussion, in which the struggling groups can benefit from hearing the ideas from other groups while getting feedback from their peers on the issues with their own ideas. Throughout the classroom experience, it is important to remember that you are rarely lecturing — your primary role is to facilitate their modeling efforts.

There may be times, however, when you do need to take the stage and teach a lesson. For example, before having students jump into a project in which they will need to look at data and fit a model to it, it may be helpful for you to lead an overview of families of functions and their graphs. As another example, if your course is structured around components of the mathematical modeling process, you may want to open each unit or chapter with a mini-lesson on that component, including offering suggestions on how to do it well and identifying common pitfalls that you hope your students avoid. Some common pitfalls we see are:

- If students are asked to find the “best” way to do something, they may be inclined to come up with some sort of score that achieves their goals both inexpensively and in the fastest time possible. In the end, they may have a cost component for their model and a time component. Inexperienced modelers might decide to simply add those components, leading to them adding a term with units of dollars to a term with units in hours.
- Students may get so involved in and excited about their model that they forget it isn’t obvious to an outsider. Students often have to be reminded that when they are presenting their models in writing, they should explain their ideas to an audience that is unfamiliar.
- The concept of making assumptions can be perplexing to students. It’s not uncommon for early modelers to include anything that seems like an assumption, whether it is important or not. For instance, in the Driving for Gas problem in Chapter 1, students might include a statement like, “We assume that there are 16 hours of daylight.” This assumption should not be included in this case because it is completely irrelevant to the model.

More suggestions and common pitfalls can be found in the aforementioned handbook, *Math Modeling: Getting Started, Getting Solutions*.

COACH

As previously noted, some students struggle with the abrupt transition from what they thought a math class should be (watching a teacher do examples for which there is a clear procedure and a singular answer, and then mimicking that procedure for a set of problems each with a singular correct answer) to what they encounter in a mathematical modeling course (being handed a challenging problem that may not even appear to have any mathematics in it, with little to no guidance other than being told they need to do it all on their own). This transition, if not properly supported, can be overwhelming to the

point of being disabling for students. Additionally, as previously noted, in the world of mathematical modeling, student decisions carry consequences in terms of the work that is done and the results of that work. The gravity of these consequences and the value placed on decisions causes the classroom to be much more emotionally charged than a typical mathematics classroom. Therefore, perhaps one of the most important roles we have as instructors is that of coach.

...one of the most important roles we have as instructors is that of coach.

Before game day, a coach explains important strategies to her team, knowing that these strategies will need to be modified to address the unique challenges that arise in every real game. On game day, the coach doesn't jump in and play the game for her team, but she is always on the sidelines, carefully observing the action. After game day, the coach reviews the game with her players and offers constructive criticism and feedback so that the team can learn from each game experience and get stronger. Similarly, we should give our students some idea of what to expect before they are even given a problem; we should overtly tell them that this course will be different and feel different from any other mathematics course they have taken, that the initial feelings of frustration are normal, and that over the course of the semester, these feelings will give way to feelings of accomplishment. During the modeling process, we should be vigilant observers of their modeling process. As students complete the modeling process, or components thereof, we should offer detailed, insightful, and timely feedback to our students. Through this feedback process, we can facilitate the development of their skills and modeling prowess throughout the semester.

A good coach also knows her players well. She knows when they will best benefit from a firm and honest assessment of a weak performance, versus when they will best benefit from a more encouraging conversation. Considering the high level of emotional investment in the setting of a modeling classroom, we also need to work to understand how to best engage with our students as they navigate the challenges of the course.

ASSESSMENT

Just as the design and implementation of a mathematical modeling course is different than for a typical mathematics course, the assessment is also different. There is no online homework system for modeling, and while it is easy to test whether or not students know the quadratic formula and how to use it on a 50 minute exam, it is not obvious how to assess the degree to which students know the modeling process and can apply it correctly.

Fortunately, there are a variety of ways to assess student learning in a mathematical modeling course, but before discussing methods of assessment, it is important to determine what you plan to assess. Remember that the goals of a mathematical modeling course are multifaceted and could include:

- Mastery of applying the individual components of the modeling process;
- Mastery of applying the entire modeling process;
- Ability to effectively and appropriately communicate findings;
- Ability to work as a member of a team;
- Development of persistence.

We briefly address reasonable means for assessing each of these in the paragraphs below, followed by a discussion of how to translate from the assessment process to feedback and ultimately a grade.

MASTERY OF INDIVIDUAL COMPONENTS OF THE MODELING PROCESS

Although this is not true for all components of the modeling process, this is one of the few course goals that can actually be assessed fairly easily in an exam setting. For example, to assess their ability to evaluate assumptions, you can provide a problem, a model, and a list of possible assumptions. Then you could ask students to identify the assumptions that are relevant to the problem or model; you could also ask students to provide justifications for those relevant assumptions. To assess students' abilities in evaluating a model, you can provide them with a problem, data, and a few mathematical models, and then ask them to rank the models and to explain the reasoning for their ranking.

MASTERY OF THE ENTIRE MODELING PROCESS

While it may be possible to have students solve a toy modeling problem within a single class period, a truly meaningful modeling problem takes time. Also, students will unravel the problem and work towards a solution at a wide variety of paces, so it would not be reasonable to force the hard work of completing the modeling process into a single class hour for an in-class exam. Furthermore, the modeling process is best accomplished in groups. Therefore, group projects may be the optimal conditions for observing your students' mastery of the entire modeling process.

ABILITY TO EFFECTIVELY AND APPROPRIATELY COMMUNICATE FINDINGS

The last component of the modeling process is reporting the results. As the instructor, you can decide how much guidance you provide on the format of the submission. If you have spent significant time discussing the best means for communicating ideas based on audience, goal, and medium, then perhaps you can make part of the assignment determining the details of the correct format, such as level of detail and voice; however, if you have not discussed this, you should probably provide your students with more guidance on their reports, particularly early in the semester. The reports, which could be group reports or individual reports, offer an excellent opportunity to assess written communication. Oral, visual, and multimedia presentations are also good ways to assess other communication skills; these products, if shared with the class, can also give students an opportunity to see how their classmates may have taken a different approach to solve the same problem. You might consider having students assess each other's presentations.

ABILITY TO WORK AS A MEMBER OF A TEAM

You may decide that this is not an important goal for your course, and therefore you may choose not to assess it. While it is difficult to develop concrete assessment tools to measure a student's team spirit, you can develop a more abstract assessment based on observations of group dynamics in class, and you can also ask students to provide feedback about their group interactions and their perceptions about working with each group member.

PERSISTENCE AND OTHER HABITS OF MIND

As with teamwork, this is difficult to assess in a concrete way, but you can make observations and provide feedback for improvement.

ASSIGNING GRADES

In mathematical modeling, there is no one right answer. Some approaches and answers are patently better than others, while two very different approaches might ultimately demonstrate similar levels of modeling competence. This can make grading difficult. There are multiple approaches to grading, and it truly depends on you, as the instructor, as to which will work best for you. Here are a few possible grading approaches for assessing student submissions:

Process rubric You could develop a rubric that identifies the key components in the modeling process, including problem-specific aspects. For each item on the rubric, write the level of competency you would expect to see from a team that did very well, from a team that did sufficient work, and from a team that did not do very well. By weighting these and assigning points to each rubric item, a numerical score can be determined.

Minimal rubric / holistic As you read each paper, you can simply decide whether their work was fully clear, convincing, and complete (A-level work), met the minimum standards of the assignment (C-level work), or failed to do well on the assignment (F-level work), with papers that fall between an A and C earning a B, etc.

Detailed rubric You could come up with a detailed rubric that addresses every step that you would expect a team to encounter in the process including the correct answers; however, in a truly open-ended problem, this rubric will not adequately allow you to evaluate the diversity of solutions you may encounter. For this reason, we strongly discourage assigning a strict number of points to components of a preconceived idea of a correct answer.

Regardless of which grading approach you choose, there are a few ideas that are worth implementing in order to improve student learning and classroom attitudes, including your own:

- Feedback is even more important for promoting growth than a grade, so be sure to provide your students with sufficient comments for them to improve on future assignments;
- Make sure to share your grading policies and schemes with your students. It is important that they understand how they are being assessed;
- Grading long, detailed, written projects can seem like it would be overwhelming, but it doesn't have to be. Come up with a strategy that helps you focus your effort and time on the things that you truly want to assess, recognizing the tradeoff between your time and the quality and quantity of the feedback you give your students.

- Providing interim feedback can help prevent your students from spending a large amount of time heading in a wrong direction or putting together an insufficient product. Opportunities to assess partially complete work could be informal, such as discussions as you walk around the classroom, or formal, such as the submission of an outline, a portion of the write-up, or the mathematics. The work that goes into grading these interim submissions can improve the final submission to the point that the total time spent grading is actually reduced.
- Not every project needs a full write-up. Students may begin to feel burned out by the writing process, and you may begin to feel burned out by grading, so consider asking only for an executive summary. Alternatively, you can opt to focus the assessment of a particular project on one or two sections of the report; if you do this, be sure to let your students know where to focus their efforts.
- Consider having a mandatory revision policy. Grading projects can be burdensome, especially if the work requires a lot of feedback. If you find that the project has significant flaws, you may consider having a conversation with the author(s) to provide your feedback in person, and then requesting them to revise and resubmit. The conversation will probably not take longer than would providing written feedback, but it gives students an opportunity to participate in the conversation about what you value in their work, making this submission, and likely those that follow, significantly better.

For additional suggestions please see Appendix D.

**MODELING ACROSS
THE STEM
CURRICULUM**

MOTIVATION

At the beginning of this section we focused on the development and implementation of a mathematical modeling course for non-STEM students, and the skills developed in such a course are similarly important for STEM majors. These skills include transferable skills such as: distilling a large, ill-defined problem into a tractable question; identifying and using reliable resources; working effectively in a group; and communicating effectively.

For STEM majors, particularly non-mathematics majors, the required mathematics courses can seem unrelated to their disciplinary and career goals, especially if the mathematics is taught devoid of meaningful and relevant applications. The engineering education community has recently issued pleas for transparency of the relevance of mathematics. Contextualizing the mathematics in the form of modeling questions helps answer this call for demonstrating relevance to students. Furthermore, learning mathematics in context has been demonstrated to promote deeper conceptual understanding of material.

For these reasons, all STEM majors should be afforded opportunities to engage with elements of the modeling process. These opportunities could come in the form of a mathematical modeling course for STEM majors or through the inclusion of modeling activities in the existing STEM mathematics curriculum; both of these options are discussed in the next section.

IMPLEMENTATION

We realize that many STEM curricula, particularly engineering programs, are already flush with credits, which could make it difficult to add a full modeling course. Therefore, the majority of this section addresses ways to fold mathematical modeling into the standard STEM mathematics curriculum, but first we briefly point out some considerations for those of you who may already have or may consider adding a mathematical modeling course for all STEM majors, not just mathematics majors.

A Mathematical Modeling Course for STEM Majors The MAA's 2015 Committee on the Undergraduate Program in Mathematics Curriculum Guide includes recommendations for the development of a mathematical modeling course for mathematics majors, but the skills addressed in modeling are also important for non-mathematics STEM majors. In considering a full mathematical modeling course for all STEM majors, there are different benefits depending on the location in the curriculum. Specifically, an early course provides students with a modeling foundation that can then be leveraged throughout the curriculum. Alternatively, a course later in the curriculum can draw upon a broader and more mature set of mathematical tools, increasing the complexity and diversity of issues that students can tackle. Ideally, mathematical modeling should be something that students see throughout their entire undergraduate curriculum, so even if an upper level mathematical modeling course exists, the STEM student curricular experience can be enriched by also folding modeling activities into lower level STEM courses, such as calculus, linear algebra, and differential equations.

Modeling in Calculus, Linear Algebra, and Differential Equations The majority of STEM majors take at least a few courses in the calculus sequence, if not the entire sequence, and many of them also take differential equations and/or linear algebra. Given the wide audience, these courses provide optimal settings to introduce students to mathematical modeling. In this section, we will present how modeling can be used to facilitate the existing learning goals of these courses by discussing: where modeling can be used in a course; what students do when they model; and the roles that you, as an instructor, have as you help students develop both their mathematical and transferable skill sets.

Using Modeling in Existing Courses Consider your first semester calculus course; the course covers a full list of topics that are taught along with specific learning objectives, and it also probably also includes applications, such as velocity, related rates, and optimization. Similarly, the remainder of the calculus sequence and other required mathematics courses for STEM majors also probably already include applications. However, it is important to remember that applications are not necessarily modeling problems. As was illustrated in Chapter 1, an application problem may be modified to become a modeling problem. A modeling problem is open-ended, giving students autonomy throughout the modeling process as they define the problem, make assumptions, find data, develop a model, test the

model, use the model, analyze the solutions, draw conclusions, and report their findings. However, the level of autonomy can be restricted such that students are able to experience many aspects of the modeling process while still addressing a particular mathematical skill. For example, you can have students collect data on the temperature of a cup of coffee cooling to room temperature. If you ask students to plot their data in the form of change in temperature versus temperature (that is, temperature, not time, is on the horizontal axis), then students will likely find that the derivative of temperature is approximately a linear function of temperature. Therefore, they will practice solving the differential equation while they engage with elements of the modeling process.

As we've just seen, you can introduce a modeling problem that has students employ a recently learned skill. Alternatively, a modeling problem, such as the coffee problem above, can be introduced at the beginning of a unit for the purpose of motivating the lessons of the entire unit, revisiting the problem frequently as they begin to build the tool set needed to address the problem.

Yet another approach is to use modeling to draw connections between ideas from different courses, such as a modeling question in a calculus course for which the solution would require some basic statistical reasoning or a modeling problem in an integral calculus course, that also has students think about rates of change, drawing on their previous experiences in differential calculus.

Here are the main ideas to keep in mind as you begin to add modeling to your existing courses:

- Applications are not necessarily modeling – there needs to be some autonomy and open-endedness.
- Even open-ended problems can still be presented such that particular curricular skills can be engaged and assessed.
- Modeling can be used:
 - > To motivate new material;
 - > To provide practice for recently learned material;
 - > To tie together disparate material.

**HOW DO STEM
STUDENTS DO
MATHEMATICAL
MODELING?**

There is a lot of flexibility in how you fold modeling into the existing curriculum. Depending on the goals of your STEM mathematics courses and the flexibility of the schedule, your students may engage in the entire modeling process, or just employ a few elements of it. They may have many small modeling questions throughout the semester, or they may have only one or two large projects. Students may work in isolation, they may work in groups, or the whole class could work together. Work on the modeling problems could occur outside of the classroom, or modeling ideas can become part of daily lessons and discussions as

the class considers how each newly taught idea impacts their model, their solution, and the interpretation of their results.

Regardless of the approaches you take and the level to which your students engage with the modeling process, it is important that students take ownership of the modeling experiences by insuring some autonomy in the process.

FACULTY ROLES

In the previous section, we presented the following four faculty roles: course developer; problem developer; classroom facilitator; and coach. As you add modeling to existing courses, you still need to consider many of the same issues, so we point you to that section for detailed explanations of these roles. Below we highlight some considerations that are not addressed in that section.

Course and Problem Developer Since the goals of your mathematics course most likely relate to the learning of mathematical content, you will need to design modeling problems that have enough information that students will most likely develop a particular type of model, such that they will use the targeted mathematical content.

Also, as the course developer, you will need to consider how much time you wish to devote to the teaching of the modeling process and transferable skills, like written communication. You will have to make a conscious decision about whether to devote class time on these (important) secondary course goals. While this does take precious class time away from discussion of mathematical content, investing time in the introduction of the modeling process and practices in writing and communication early in the semester can increase students' productivity over the course of the semester.

Coach STEM students are likely to encounter modeling problems in their careers, and the skills they learn in a modeling class are important and useful. However, you may get even more resistance about mathematical modeling from STEM students than from non-STEM students because many STEM students have thrived in the traditional mathematics classroom, in which they simply had to identify problems by type and apply an algorithm to it. Modeling problems push students to think, to make choices, and to accept the consequences of those choices, including the possibility of failure, and this can be extremely uncomfortable for students who see themselves as good at mathematics and have not experienced challenge or failure in their previous mathematics courses. It is important that this group of students is overtly told that the modeling process often includes the experience of failure such that even good models can be improved, leading to better insights and answers of the real world problem at hand. At this time in our history when the retention of STEM students has received Presidential attention, it is imperative that we embrace the role of coach, giving our STEM students the encouragement they need to embrace these challenges and learn the joy of overcoming failure.

ASSESSMENT

Unlike the assessment in an undergraduate course on mathematical modeling, the focus of assessment in a content-specific course, such as calculus, should primarily be on the content. While you may still wish to assess the modeling process and transferable skills (e.g. communication), as discussed in the previous section, in a content-driven course, you will also need to assess your students' mathematical content skills, such as their ability to differentiate and integrate. Mathematical modeling projects need not be supplemental assignments that are tacked onto a course already filled with graded homework, quizzes, and exams, but rather, they can be used as meaningful tools to help us, as instructors, gain a more complete assessment of our students' mastery of the material. Through their development, usage, and interpretation of mathematical models and results, our students showcase both the mechanics of the course material as well as their level of understanding behind the material; this affords us a rare opportunity to assess students' ability to apply the mathematical content of our courses to real world problems.

REVISITING THE GUIDING PRINCIPLES AND CONCLUSIONS

As we close this, we feel it would be helpful to revisit the guiding principles from Chapter 1 and view them through the lens of teaching mathematical modeling at the undergraduate level.

MODELING (LIKE REAL LIFE) IS OPEN-ENDED AND MESSY.

As faculty teaching undergraduates, we are charged with transforming young people into the contributing members of society who are prepared to tackle the open-ended and messy challenges of the real world.

WHEN STUDENTS ARE MODELING, THEY MUST BE MAKING GENUINE CHOICES.

Given the diverse interests, knowledge bases, and life experiences that undergraduates bring into the classroom, groups of our students are ready to offer rich insights into modeling challenges, as long as we support them.

START BIG, START SMALL, JUST START.

If you are developing a whole modeling course, perhaps you are wondering how you could start small; before investing significant energy in launching a whole new course, you could first try one or more modeling activities in your existing courses. However, you can also just jump into the modeling experience with your students, learning together as you go.

ASSESSMENT SHOULD FOCUS ON THE PROCESS, NOT THE PRODUCT.

As we assess our students' work, we want to make sure that our goals for the course and the assessment metrics we use are aligned with one another; we also need to insure that we clearly communicate our expectations as they relate to the goals and assessments so that our students are set up to thrive in this nontraditional course setting.

MODELING DOES NOT HAPPEN IN ISOLATION.

As we prepare students to enter the work force or academia, they should be prepared to work well in teams. Additionally, it is important that students understand how to leverage the work of others through research and citation, so that they are prepared to make meaningful contributions to the world.

5



WHAT IS MATHEMATICAL MODELING, THE ART AND FLAVOR

INTRODUCTION

When the GAIMME project first began we asked the writing team of experienced mathematical modelers, mathematicians and mathematics educators for their own views on what modeling is and what it isn't. We present excerpts of these here, in order to hopefully add a richer flavor to our description and to expose how modeling is viewed by practitioners.

Dr. Henry O. Pollak was President of the Mathematical Association of America and Assistant Director of Mathematical Research at Bell Labs and Bellcore. His comments, taken from *A History of the Teaching of Modeling, History of School Mathematics, Vol. 1, NCTM, 2003*, focus on the relationship of mathematics and the real world:

“Mathematicians are in the habit of dividing the world into two parts: mathematics and everything else, sometimes called the ‘real world’. People often tend to see these two as independent of one another — nothing could be further from the truth. When you use mathematics to understand a situation in the real world, and then perhaps use it to take action or even to predict the future, both the real world situation and the ensuing mathematics are taken seriously. The situations and the questions associated with them may be any size from huge to little. The big ones may lead to lifetime careers and whole university departments may be set up to prepare people for such careers. At the other end of the scale, there are small situations and corresponding questions although they may be of great importance to the individuals involved: planning a trip, scheduling the preparation of Thanksgiving dinner, hiring a new assistant or bidding in an auction.

“Whether the problem is huge or little, the process of ‘interaction’ between the mathematics and the real world is the same: the real situation usually has so many facets that you can't take everything into account, so you decide which are the most

important and keep those. At this point, you have an idealized version of the real world situation, which you can translate into mathematical terms. What do you have now? You have a mathematical model of the idealized question. You then apply your mathematical instincts and knowledge to the model and gain interesting insights, examples, approximations, theorems and algorithms. You translate all of this back into the real world situation and hope to have a theory for the idealized question. But you have to check back: are the results practical, the answers reasonable, the consequences acceptable? If so, great! If not, take another look at the choices you made at the beginning, and try again. This entire process is called mathematical modeling.”

Throughout this document we have tried to point out the **distinction between a mathematical model and the process of mathematical modeling**. As Dr. Katie Fowler of Clarkson University has so eloquently put it:

“A mathematical model is a tool that uses mathematics to represent a real-world situation to facilitate quantification, analyses, predictions, and gaining insight. The process of mathematical modeling is the creation of such a tool and often requires significant creativity and brainstorming, but not necessarily complicated mathematics. The process itself includes determining what the output of the model is (what is being quantified?) and identifying what variables may impact that quantity. Assumptions must be made to determine relationships between the input and output of the model. Ultimately there may be multiple mathematical models for any given scenario with varying degrees of complexity. The model itself depends on the available resources and knowledge of the person creating it. The modeling process includes validating the model to assess how well it performs at making a prediction or measurement and the consideration of refinements.”

A way to think about the steps of the modeling process and why they are important is provided by Dr. Rachel Levy of Harvey Mudd College:

“Effective mathematical modeling:

- Uses mathematical tools (defined broadly) and chooses the methodology to suit the problem.
- Makes sense of something in the world (answers a question, discovers/explains an underlying phenomenon, makes predictions/connections)
- Involves iteration to create and improve the model as needed based on available evidence.
- Balances feasibility and simplicity with optimality by making useful approximations.
- Communicates the model and/or the results effectively for the client/audience.

Dr. Joseph Malkevitch, York College (CUNY), states:

“Mathematical modeling is the branch of mathematics which deals with the use of mathematics to get insights into fields and situations outside of mathematics. ...A major advantage of using a mathematical modeling framework throughout K-12 mathematics is that it provides a rubric for showing students a view of mathematics which incorporates many other desirable ideas from the point of view of society and of the mathematics community, without making mathophobia such a common outcome of K-12 mathematics.”

**MODELING VS
MODEL**

Dr. Frank Giordano (Based on A First Course in Modeling, First Edition, Giordano and Weir), Brooks Cole, 1985

“So what is *modeling*? First, it is a process to connect real-world systems (observed behavior or phenomenon) with the mathematical world (models, mathematical operations and rules, and mathematical conclusions.) A system is an assemblage of objects joined in some regular interaction or interdependence. The modeler is interested in understanding how a particular systems works, what causes change in the system, and the sensitivity of the system to certain changes. Often, the modeler is interested in predicting what changes might occur and when they occur. At the heart of the process is the construction or selection of a model.

“So what is a *model*? A mathematical model is a construct designed to study a particular real-world system or phenomenon. The construct may be graphical, symbolic, a simulation, or an experimental construct. Given a phenomenon of interest we may choose a mathematical representation, or we may choose to replicate the behavior.”

From Dan Teague, North Carolina School of Science and Mathematics (NCSSM), the world, through a mathematical lens...

“A mathematical model is a mathematical caricature of some process or phenomenon of importance and interest in our lives. This caricature is drawn with mathematical concepts and symbolism, rather than charcoal or water colors, and, like all caricatures, emphasizes some aspects of the real phenomenon and diminishes or ignores others. The method of intentional ignorance, in which some details of the real phenomenon are ignored or assumed to be well-behaved, allows us to keep the representation simple and uncluttered. The modelers strive to use temporal ignorance effectively, with some initially ignored components being added back into the model once it begins to take shape.

“In the first pass, modelers devise a mathematical representation which captures the simplest form of their problem while retaining the essence of the problem. They then probe and poke that representation with mathematical tools until it yields some understanding of or insight into the phenomenon being investigated that was not evident before the model was created.

“Through simplification, the modelers hope to capture some important aspects or structures of the phenomenon in mathematical form and, by applying the standard rules of mathematics creatively, come to view the process or phenomenon from a new perspective from which the needed insight and understanding may be gained. This new understanding or insight gives the modelers a new starting point for the next iteration of their model, in which some assumptions can be relaxed and the model captures more of the phenomenon and becomes less caricature and more portrait.”

Karen Bliss, Virginia Military Institute (VMI), describing phenomena with math:

“I see mathematical modeling as being a response to a question or a desire to describe some phenomenon. Perhaps one wants to know the “best” way to design something. Or perhaps one has run across some data and would like to be able to explain it, or even make a prediction about the future values.

“Once a question is posed, we hope to quantify aspects of the phenomenon. Perhaps we consider laws of physics or we take into account our knowledge of another discipline. For example, we might think about how exactly disease spreads (infected person coming in contact with a susceptible person) in order to quantify transmission rate.

“Ultimately, our mathematical model is an equation or equations that we believe describe the phenomenon. At this point we have moved from the physical phenomenon into the mathematical realm, and we can use the wealth of tools mathematics provides to help us find a solution to the mathematical problem at hand. (Note: The term solution isn’t well-defined term in this context since the type of solution depends on what question we asked; it may be a number, it may be a function, it may be a vector of numbers represented the value of a quantity over the next ten years, it may be a statistic, etc.)

“The solution we find is a solution to the math problem. We have to consider whether this mathematical solution makes sense in the context of the physical phenomenon. Does it fit the data? Can we trust it to explain the phenomenon? Did we capture the important features of the physical phenomenon, or did we dismiss something?”

And so modeling is a *process that tells us about the world*, as Dr. Rose Mary Zbiek of Penn State University emphasizes:

“Mathematical modeling is a process of empirically or theoretically producing one or more mathematical entities that, with its/their properties, captures targeted aspects and relationships among aspects of a real or imagined situation, ideally done by the modeler in service of solving a problem or garnering an insight related to the situation.”

When we model the real world we are always making an approximation of reality and getting an approximate answer when we are done. This is the difference between being “definitive” and being “defensible.” While there will seldom be a right way or a right answer, we must be able to defend the choices we make in terms of what we understand about the real-world situation and by what we are able to do mathematically. But modeling is often a two-way street, as Landy Godbold of the Westminster Schools (retired) points out:

“Throughout the process of modeling a primary goal is to use information from one point of view to guide work in the other. Modeling typically begins in the “contextual” world, with the modeler making decisions regarding which features of the situation are thought to be essential and which might be simplified or ignored initially in order to construct a tractable but meaningful mathematical representation. Being able to play ‘what if’ by varying parameters within a mathematical construct can replace a great deal of physical experimentation, for example. Modeling may take the modeler from ‘the answer’ to a world of answers.”

Building models requires creativity, understanding of mathematics and contexts, and the artist’s ability to accept (embrace) representations that are necessarily incorrect in some aspects in order to gain insight. Modeling involves simplification. Which aspects of a situation to simplify and which to try to represent more exactly are decisions that can vary. Such decisions depend on understanding of the context, on the objectives of the model, and on the mathematical tools available to the modeler and, perhaps, a “consumer” of the model.

“Students hear about models every day. Models predict the paths of storms. Models describe the course of a disease such as Ebola. They help shape the political landscape. From an educational standpoint, “doing modeling” provides an avenue by which students may move from “playing scales” to ‘making music,’ to borrow an idea expressed by Dan Teague many years ago.”

Since mathematical modeling is a process it is important to *distinguish it from mathematical applications*. As Pollak has written:

“Every application of mathematics uses mathematics to understand, or evaluate, or to predict something in the part of the world outside of mathematics. What distinguishes modeling from other forms of applications of mathematics are (1) explicit attention at the beginning to the process of getting from the problem outside of mathematics to its mathematical formulation and (2) an explicit reconciliation between the mathematics and the real world situation at the end. Throughout the modeling process, consideration is given to both the external world and the mathematics, and the results have to be both mathematically correct and reasonable in the real world context.”

In most educational settings, applications are introduced at the end of chapters teaching specific techniques. Students are well aware that they are meant to use those techniques to solve the given application problem. But in a typical mathematical modeling problem, the problem does not announce what mathematics we should use to analyze and/or solve it. In fact, as problems come from the real world, it may not be immediately clear that mathematics can be of help at all. This issue of how to begin work on a modeling problem, i.e. where to start, is one that makes the teaching and learning of mathematical modeling so challenging and ultimately so rewarding. We dedicate this report to furthering that effort.

A



APPENDIX A: MATHEMATICAL MODELING RESOURCES

INTRODUCTION

In this section, example problems from various different resources will be given with a three-fold purpose: first, to inform the reader about the types of modeling problems available in common resources, second, to help the reader identify how, exactly, the mathematical modeling process is addressed in these example problems, and third, to help the reader identify ways some mathematical modeling tasks can be modified to emphasize different parts of the modeling process that the given task may not address.

STATE-LEVEL EXAMS

There are two main consortia charged with developing assessments aligned with the Common Core State Standards in Mathematics (CCSSM): the Partnership for the Assessment of Readiness for College and Careers (PARCC) and Smarter Balanced Assessment Consortium (or simply, “Smarter Balanced”). As such, it is reasonable to look toward these exams and how modeling is assessed within them in order to gain an understanding of expectations for modeling competencies. It should be noted, however, that the assessment of modeling is in its infancy. This means that there are various levels of success with regard to ensuring that modeling competencies are actually assessed.

At time of writing, PARCC has released few examples of the types of modeling problems students will encounter. In a typical PARCC modeling item, the problem is specified, which does not allow the student to show their ability to identify the problem to be solved. Further, assumptions rarely need to be made, and when they are made, the student may easily complete the problem without recognizing this fact. Important variables are usually explicitly given. Model analysis and assessment and iteration are also largely absent. It is common, however, for students to report their results, including some explanation of their solution method.

PARCC EXAMPLE PROBLEM: ART TEACHER'S RECTANGULAR ARRAY

The modeling items released by PARCC indicate a preference toward scaffolding to help the student proceed through each task. The Grade 3 modeling item “Art Teacher’s Rectangular Array” consists of three questions. The overall goal of the modeling item is not stated or left for the student to identify at the outset of the problem. The item is introduced to the student with a simple explanation: “An art teacher will tile a section of the wall with painted tiles made by students in three art classes.” The number of tiles from each class is given.

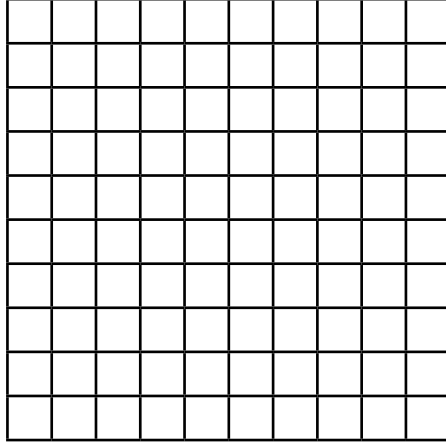
Part A The student must determine how many tiles in total the art teacher has with which to tile the section of the wall. This step is where essential variables have been defined albeit by the writer of the item, not by the student. The variable in question is the total number of tiles.

Part B A 10×10 grid is given (see Problem A.1). The student is instructed that a rectangular array must be made from the tiles and that each box of the grid should represent only one tile. Here, again, variables are defined for the student, namely how many tiles can fit and are reasonable to be fit in a certain space and how tiles must be arranged. The student is instructed to shade the squares of the grid to show how a rectangular arrangement may be made from the all the tiles. Note, a student may arrive at a solution through trial and error, despite the goal of the assessment being for the student to make use of the relationship between multiplication and rectangular arrays.

Part C Part C explains to the student that another rectangular array is made using 56 tiles with 7 tiles per row and that “a multiplication equation using R to stand for the number of rows [is] used.” The student is instructed to write that equation using R . This part requires the student to extend the model created in Part B to a “more” mathematized model and solution than was necessarily created in Part B. In the Grade 3 example, some of the steps the mathematical modeling process are involved (do the math, iterate), but not nearly all of them. This problem could be adapted for classroom use by allowing students to determine the constraints of the situation on their own, for example. The teacher could lead a discussion regarding how best to arrange tiles or may even indicate a preference toward a rectangular arrangement. Upon agreeing on a rectangular arrangement (however it comes about), the teacher might then ask students to make a rectangular arrangement of the tiles. Many different arrangements might be presented. The students might then be asked to implement the model on a classroom wall. They may find that some of the arrangements don’t fit; this will lead the students to discover that the space available also matters, as well as whether or not overlap is allowed or if spacing should be placed between the tiles. Once the students account for these constraints, they can then generalize the model.

The grid shows how much wall space the art teacher can use. Use the grid to create rectangular array showing how the art teacher might arrange the tiles on the wall.

Select the boxes to shade them. Each tile should be shown by one shaded box.



PROBLEM A.1: PARCC GRADE 3 MODELING ITEM, "ART TEACHER'S RECTANGULAR ARRAY," PART B

PARCC EXAMPLE PROBLEM: TEMPERATURE CHANGES

A PARCC Algebra II/Math III modeling item shows evidence of assessment of some of the parts of the mathematical modeling process that are not present in many other assessment items, particularly making assumptions, defining essential variables, and assessment of the model. In "Temperature Changes" (see Problem A.2), the scenario is introduced by explaining that a scientist is studying cooling patterns as a substance cools from 200°C to 0°C. A chart with the temperature of a certain substance at 0, 40, 80, and 120 seconds into the experiment is given. Further, a graph showing time versus temperature is given, these points are plotted, and three possible models, curves A, B, and C, are fitted onto the graph.

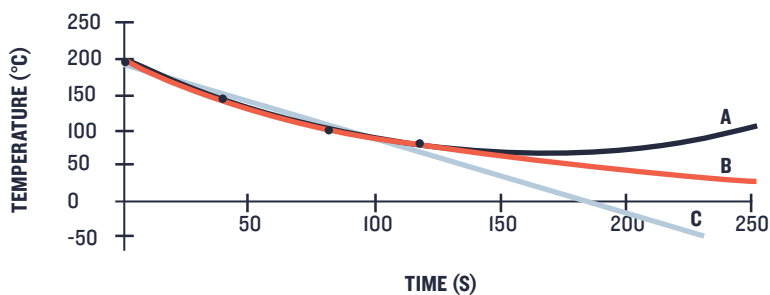
Part A The student must match the type of model to the actual curves (the choices are linear, quadratic, and exponential), identify the best model for how the substance cools over the 0 to 250 second time period, and must explain why the other models don't fit the data as well. This item requires that assumptions be made, specifically that the trend seen in the four given points will continue. In choosing the best type of model, the student must determine essential variables in the scenario, specifically that the temperature should not increase (thereby eliminating the quadratic model) and that the temperature, by design, cannot fall below 0°C (thereby eliminating the linear model). The item also requires that the student assess the models presented to determine which is best and to explain their assessment.

Part B The student is tasked with constructing an equation for the specific model. This is the familiar task of doing the math. It should be noted that in order to complete this part of the problem, the student must make use of estimation, as the base of the exponential function is not the same for each successive temperature change. The bases are all close to one another, so in using this model, the student must accept that the model will not perfectly represent the cooling of the substance in order to complete the problem successfully. If used in a classroom setting, it would be reasonable to discuss explicitly the assumptions that go into the solution to this problem.

A scientist is studying the cooling patterns of a particular material over time. Her research requires heating a sample of the material at 200°C. She records the temperature of the sample as it is cooled to 0°C. The table shows the data collected during the first 2 minutes of the cooling process.

TIME MATERIAL IS COOLING (SECONDS)	0	40	80	120
TEMPERATURE (°C)	200	141	101	74

The figure shows the scientist’s data (data points are plotted as large dots). Three possible models for the data are also shown: a linear model, a quadratic model, and an exponential model.



Which model is linear? Which model is quadratic? Which is exponential?

Which model is best for the range of times $0 < t < 250$?

Explain why the other models do not fit the data very well for the range of times $0 < t < 250$.

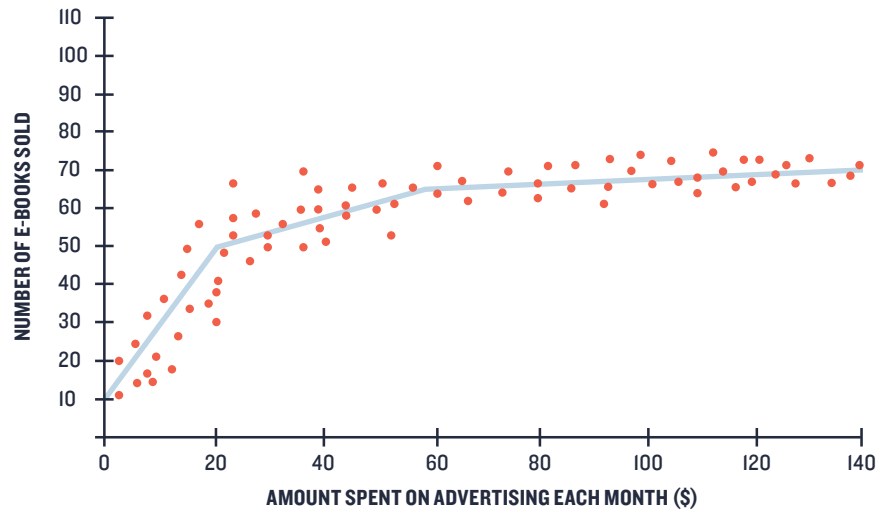
PROBLEM A.2: PARCC ALGEBRA II/MATH III MODELING ITEM, “TEMPERATURE CHANGES”

SMARTER BALANCED EXAMPLE PROBLEM: E-BOOK ADVERTISING

Smarter Balanced features both mathematical modeling and problem solving items. Of the released items, there are many more examples of problem solving than of modeling. However, the modeling items that have been released show a tendency toward introducing one significant problem without scaffolding the task for the student. It should be noted, however, that this is based on only two sample modeling problems, both aimed toward the high school level.

One of the two modeling problems released revolves around eBook sales. The student is presented with a scatter plot that is broken into three portions, each having its own line-of-best-fit (see Problem A.3). These three lines are connected. The plot shows the relationship between the amount of money someone named Tyler spends on advertising and how many eBooks are sold in turn.

Tyler earns \$3.00 for every e-book he sells on his website. (E-books are books that are available electronically.) He investigated the relationship between the amount spent on advertising each month and the number of e-books sold. He used this information to determine the lines of best fit shown in this graph.



What is the greatest amount Tyler should spend on advertising each month? Show your work or explain how you found your answer.

PROBLEM A.3: SMARTER BALANCED ITEM 43028 ON E-BOOK ADVERTISING.

The introduction to the scenario explains that Tyler earns \$3 for each eBook sold. The student must determine how much money Tyler should spend on advertising each month. This item requires the student to identify the problem to be solved, to make assumptions, to use the model to get a solution, and to report the results. In order to solve the problem, first, the student must identify precisely the problem to be solved. With some calculation, the student must determine that for $0 \leq x \leq 20$, Tyler earns \$6 for every dollar spent on advertising, for $20 \leq x \leq 60$, Tyler earns \$1.13 for every dollar spent on advertising, and for $x \geq 60$, Tyler only earns \$0.94 per dollar spent on advertising, where x is the amount spent in dollars. This means that one student may interpret the problem to be to maximize profit outright, while another student may interpret the problem to be to make a decent profit while maintaining some level of safety in investing due to the variability of the actual

profits. In solving the problem, the student must also (implicitly) make several assumptions, for example, that the trends shown will continue and that Tyler has enough money to spend on advertising. Once the problem has been identified and assumptions made, the student must use the model to arrive at some solution that satisfies those decisions. Upon arriving at that decision, the student must report on the process used in order to arrive at that conclusion. This problem is a fairly complete modeling problem and likely does not need much adaptation to be used in a classroom setting.

**INTERNATIONAL
COMPARATIVE
EXAMS**

The Programme for International Student Assessment (PISA) and the Trends in International Mathematics and Science Study (TIMSS) are international comparative exams that include a wide variety of problem solving activities within their mathematics portions; when possible, the problem solving problems are set in a real-world scenario, and thus, have the potential to make good candidates for inclusion in a mathematical modeling setting.

PISA EXAMPLE PROBLEM: SAILING SHIPS

One question from the 2012 PISA exam includes an interesting mathematical modeling setting. In this question, from the series “Sailing Ships” (see Problem A.4), the student is asked to determine how long it will take for a “green” investment in a kite sail, which reduces diesel fuel use, to cover the cost of purchasing it through savings in fuel. All of the relevant information is given, including the average yearly diesel fuel consumption, the average cost of diesel fuel, and the cost of the kite sail. “Distractor” figures are also included, such as the length and breadth of the sailing ship, its load capacity, and its maximum speed. Thus, the student must choose the most important variables from a list of potential variables as well as some distractor figures. In this question, the problem to be solved is quite clear: determine the value of the annual fuel savings and how many years it will take for those savings to sum to a value greater than the cost of the kite. Assumptions must be made, but, as frequently occurs in these types of problems, the student may be unaware that such assumptions are necessary to proceed. For instance, one must assume a constant diesel fuel price. This is not a particularly reasonable assumption, especially given that the kite will pay for itself only after about 8.5 years under these assumptions. Fuel costs are very likely to increase over that time span. However, this assumption will lead to a working estimate, and, if one were so inclined, a refined model might account for the increasing costs of fuel. Further, one must assume that the fuel savings will be constant, the kite will not need to be serviced at any cost and that the kite will be functional for the length of time it is installed in order for it to reduce the fuel consumption. A student certainly could solve the problem just from the information provided without recognizing that these assumptions must be made, but they are made regardless. If such a problem were to be used in a classroom setting, it would be beneficial to make these assumptions clear and, perhaps, to iterate the process in order to refine the initial model that makes use of these assumptions in order to arrive at a better, more predictive model.

Due to high diesel fuel costs of 0.42 zeds per litre, the owners of the ship New Wave are thinking about equipping their ship with a kite sail.

It is estimated that a kite sail like this has the potential to reduce the diesel consumption by about 20% overall.

NAME	New Wave
TYPE	Freighter
LENGTH	117 metres
BREADTH	18 metres
LOAD CAPACITY	12,000 tons
MAXIMUM SPEED	19 knots
DIESEL CONSUMPTION PER YEAR WITHOUT A KITE SAIL	Approximately 3,500,00 litres

The cost of equipping the New Wave with a kite sail is 2,500,000 zeds.

After about how many years would the diesel fuel savings cover the cost of the kite sail? Give calculations to support your answer.

PROBLEM A.4: PISA (2012) ITEM, "SAILING SHIPS," QUESTION 4.

PISA EXAMPLE PROBLEM: CLIMBING MOUNT FUJI

Another PISA 2012 question, from the series "Climbing Mount Fuji," also includes aspects of the mathematical modeling process that may be useful in the classroom (see Problem A.5). In the question, the student is told that a particular hiking trail on Mount Fuji is 9km long to the top. The student is given the length of the trail (including the length of a round trip), the speed at which Toshi, a hiker, estimates he can walk up the trail, Toshi's estimate for the speed he can walk down the trail, the time the trail closes, and the fact that these speeds take into account breaks. The student is asked to determine the latest Toshi can start hiking up the trail, given that he must return by closing time at 8pm. This problem, as presented, does not require the student to participate in several of the mathematical modeling processes, save for doing the math and formulating a previously-learned model. However, it could be modified easily into a very nice modeling problem for classroom use. Its strength in being a good candidate for a classroom modeling problem is its real-life question: "what is the latest time [Toshi] can begin his walk so that he can return by 8 pm?" This is a type of question that most people deal with in their everyday lives. A similar problem is "what time do I have to leave to get the bus so that I arrive at my destination on time for my appointment?" Posing the problem as it has been posed, rather than simply requiring students to calculate a total travel time ("how long will a round trip take?") is what distinguishes this problem as being

better suited for mathematical modeling. Stated this way, the problem has a real-life purpose besides simply employing a known formula. This question could be adapted by asking students to identify and even estimate the relevant variables and to make assumptions about the hike on their own. An important piece of the problem is in recognizing that the speed ascending the mountain will usually be faster than the speed in the descent. As Mount Fuji is a tourist destination, the students might want to account for time for a tourist to stop and take pictures. Withholding the numbers and assumptions from this problem will help convert this into a very realistic modeling problem. Another modification may be to provide students with a set of three different trails on which the hiker can return. Each might be different lengths (some are more winding than others) and the difficulty of the trail would imply different average speeds for descent. Which of the three return trails should the hiker take in order to return home as quickly as possible?

The Gotemba walking trail up Mount Fuji is about 9 kilometres (km) long.

Walkers need to return from the 18 km walk by 8 pm.

Toshi estimates that he can walk up the mountain at 1.5 kilometres per hour on average, and down at twice the speed. These speeds take into account meal breaks and rest times.

Using Toshi's estimated speeds, what is the latest time he can begin his walk so that he can return by 8pm?

PROBLEM A.5: PISA (2012) ITEM, "CLIMBING MOUNT FUJI," QUESTION 2.

TIMSS EXAMPLE PROBLEM: RECIPE FOR THREE PEOPLE

A Grade 4 question from TIMSS provides a realistic setting and a non-modeling question. In this question, the student is provided with a 3-item list of the quantities of ingredients (eggs, flour, and sugar) needed for a particular recipe (see Problem A.6). The student is told the recipe yields servings for 6 people. The student must determine how much of each of flour and sugar are needed for a recipe that serves 3 people. The number of eggs is given. This problem, as presented, is not a strong modeling problem. The student must recognize that reducing the number of servings requires a proportional change, and perhaps the student must assume that each of the ingredients requires the same proportional change. However, the student is presented with one solution already (the 4 eggs required in the recipe for 6 is reduced to 2 eggs in the recipe for 3 people). In this sense, the student might only need to find a pattern in how the number of servings changes and how the number of eggs changes, then employ this result for the flour and sugar. While a valuable exercise, this is not in itself modeling. Here, again, this problem could be adapted to address more

of the modeling process. In a classroom setting, presenting the students with a recipe for 6 people and asking them to determine how to use the recipe for various different numbers of people, for example, 12, 24, 3, and 2. Actually providing a recipe for the students to “cook” (for example, homemade bouncy balls¹ might allow students to identify the problem at hand: we have a recipe for so many people, but we have more, or fewer, people than the recipe will work for. Allowing the students to work with the recipe will help them make a major assumption, specifically that everyone should get the same size portion of the recipe. A demonstration of the recipe for differently-sized groups of people will allow the students to do the math both physically, and then ultimately, abstractly on paper. At this point, the students will be able to analyze their solution and their model. If it needs to be refined for some reason, it can be at this point. Finally, the students will report on their results, perhaps even making the recipe for all the students in lower grades.

The ingredients below are used to make a recipe for six people. Sam wants to make this recipe for only three people. Complete the table below to show what Sam need to make the recipe for three people. The number of eggs he needs is shown.

EGGS	4
FLOUR	8 cups
MILK	1/2 cup

EGGS	2
FLOUR	_____ cups
MILK	_____ cup

PROBLEM A.6: TIMSS (2011) GRADE 4 ITEM M031183, “RECIPE FOR THREE PEOPLE.”

TIMSS EXAMPLE PROBLEM: COST IN ZEDS FOR A TAXI TRIP OF N KM

A Grade 8 question from TIMSS, like the Grade 4 question above, is not stated as a mathematical modeling problem, but the context of the problem is taken from real life and, thus, the context can be modified into a mathematical modeling problem for the classroom. The problem is a multiple choice problem requiring students to identify the correct algebraic expression to determine the cost of an n kilometer taxi ride with a base fare of a base fare of 25 zeds (zeds are a common, invented unit of currency found in international exams) and a standard rate of 0.2 zeds per kilometer driven (see Problem A.7). As it stands, the student must simply choose the correct model (formula) from the four options given. However, these models fail to account for various surcharges and tips with which students might be aware are charged in a taxi trip. New York City taxi metered fare, for example, has different surcharges depending on the time of day, the final destination of the trip, traffic conditions, and if a toll was encountered, and, finally, a tip of between 20% and 30% is expected (see Figure A.1). Clearly, for New York City cabs, a simple model using the base rate of \$2.50 and the \$0.50 per 1/5 mile will not be sufficient to determine the actual cost of the trip. However, it would be an excellent starting point from which to account for all of the extra surcharges.

Thus, the problem could be introduced to students in the following way: you are planning a weekend trip to New York City (or any convenient city with a sufficiently convoluted fare scheme) and would like to know how much to expect to pay in cab fare. How can you predict how much you'll pay in traveling between different locations within Manhattan? Stated this way, it gives the opportunity for students to identify the major problem at hand: the length of the trip and the time of day affect the cost of cab fare. At this point, the students may make assumptions and define their variables. For example, an initial model might assume there are no surcharges, tolls, the traffic moves freely, and no tip will be given. The resulting model will be a very basic one that accounts only for the base fare and cost per $\frac{1}{5}$ mile. The students will need to assume that their typical trip will be of a certain distance and at that point, the problem can be solved to obtain an answer. Assessment of the model will show that this fare is the minimum expected cost and iteration is needed to gain better insight into the cost of taxi fare. In the least, multiplying the entire equation for the model by 1.2 will result in the expected cost once the driver's tip is taken into account. Other refinements can be accounted for by finding the most expensive time of day to travel and assuming a stand-still traffic jam for some amount of time. This will result in a model that gives a high estimate of the cost so that the students can be prepared for such a circumstance. A report of how the model was built and what charges to expect can be made, perhaps in the context of a travel budget.

The taxi company has a basic charge of 25 zeds and a charge of 0.2 zeds for each kilometer the taxi is driven. Which of these represents the cost in zeds to hire a taxi for a trip of n kilometers?

- A. $25 + 0.2n$
- b. $25 \times 0.2n$
- C. $0.2 \times (25 + n)$
- D. $0.2 \times 25 + n$

PROBLEM A.7: TIMSS (2011) GRADE 8 ITEM M032577, "COST IN ZEDS FOR A TAXI TRIP OF n KM."

**MATHEMATICAL
MODELING
CONTESTS**

Three mathematical modeling contests of note exist that challenge students to participate in extremely realistic mathematical modeling activities. These contests are Moody's Mega Math Challenge (M³ Challenge), open to 11th and 12th graders in the United States; the High School Mathematical Contest in Modeling (HiMCM), open to high school students internationally; and the Mathematical Contest in Modeling (MCM) and Interdisciplinary Contest in Modeling (ICM), each open to high school and undergraduate students internationally. Each of the contests requires students to engage in all aspects of the mathematical modeling process, including preparing extensive written reports of their rationale in solving the problem.

METERED FARE INFORMATION

- Onscreen rate is 'Rate #01 – Standard City Rate.'
- The initial charge is \$2.50.
- Plus 50 cents per 1/5 mile or 50 cents per 60 seconds in slow traffic or when the vehicle is stopped.
- In moving traffic on Manhattan streets, the meter should “click” approximately ever four downtown blocks, or one block going cross-town (East-West).
- There is a 50 cent MTA State Surcharge for all trips that end in New York City or Nassau, Suffolk, Westchester, Rockland, Dutchess, Orange, or Putnam Counties.
- There is a 30 cent Improvement Surcharge.
- There is a daily 50 cent surcharge from 8pm to 6am.
- Passengers must pay all bridge and tunnel tolls.
- Your receipt will show your total far including tolls. Please take your receipt.
- The driver is not required to accept bills over \$20.
- Please tip your driver for safety and good service.
- There are no charges for extra passengers or bags.

FIGURE A.1: NEW YORK CITY TAXI AND LIMOUSINE COMMISSION OUTLINE FOR METERED FARES (AS OF AUGUST 2015).

Moody’s Mega Math Challenge (M³ Challenge) features problems that are often quite contemporary in nature, addressing issues that can be found within the context of current events and concerns. Past M³ Challenges have included examinations of Social Security, the Stimulus Act, and energy independence. The 2015 problem addresses the true financial cost of choosing to attend or not to attend college: “What is higher education really worth? Can modeling show the cost, return, and value of a college education?” The crux of the problem lies in determining if the lost wages accrued while attending college and not working and accumulating debt from cost-of-attendance and other living expenses will be outweighed by the presumably higher wages associated with earning post-high school degrees.

The 2015 champion team in the M³ Challenge, for example, solved the college problem using three iterations of a further and further refined model. To create the initial model, the students identified the variables that must be taken into account in order to determine the lifetime cost of college, which includes not only the cost of tuition and fees and their associated loans, but also accounts for expected family contribution and loss of wages while attending college full-time (opportunity cost). The students made assumptions based on real-world data including identifying the typical age of oldest parent (which affects expected family contribution), choosing to use median data household income and assets rather than mean data, the proportion of household assets that are contributed to the expected family

contribution calculation, and the proportion of student aid that comes from loans rather than grants. From these (and other) assumptions, they made an initial model for the annual cost of college that accounts for expected family contribution, loans, and opportunity cost. After determining that their model was reasonable, they revised the model to consider the 20-year outlook based on degree held (4-year STEM, 4-year non-STEM, Associate's, and High School), cost of college, and potential earnings for each of those degrees. A third iteration of the model was then made which accounted for life-satisfaction associated with each of the degrees in order to help high school students determine not only which degree will provide them with more money, but which will provide them with the most satisfying life. Finally, after reporting on the strengths and weaknesses of their final model, the winning students provided a detailed report of their process along with recommendations for high school students based on the results of implementing their model.

The problem given by the M³ Challenge, as can be seen, requires that students address all aspects of the mathematical modeling process. The solution provided above is an example of what advanced mathematical modeling students are capable of achieving; this is not to say that these problems cannot be used as a classroom resource for use with students who are not advanced modelers. Because the problems are open not restricted to using a certain type of mathematics, students of various levels can complete them. This is the case not just for the M³ Challenge, but also the similar mathematical modeling contests HiMCM, and MCM/ICM.

NOTABLE CONTRIBUTIONS

Large-scale examinations are not the only resources one can look toward in order to find material to use in the mathematical modeling classroom. There are several organizations focused on applied mathematics and mathematical modeling with respect to their teaching, learning, and assessment. The following is by no means an exhaustive list of examples from these organizations, but is intended to give an overview to those types of works that are available.

THE BALANCED ASSESSMENT IN MATHEMATICS PROJECT (BAMP)

The Balanced Assessment in Mathematics Project (BAMP) was a project based at the Harvard School of Education from 1993 to 2003. From this project, an expansive library of tasks for K–12 was developed with modeling being one of the mathematical processes with which many of those tasks were concerned.

Stack 'em Up is one of the primary school (K–2) tasks developed by BAMP that makes use of modeling. In the task, the students are introduced to some words and phrases used for comparisons such as “smaller,” “bigger,” “taller,” “the same,” “almost as tall,” and “much bigger.” As a warm-up activity, students must twice compare two sets of chosen objects found in the classroom. The main task is largely an exercise in estimation. The first question

asks students to imagine stacking all the mathematics books in the classroom. The students must determine if the stack will be taller than the tallest student in the class and then must determine if it will be taller than the student's desk. The second question asks students to gather and stack all the mathematics books in the classroom and to make the correct comparisons between the stack and the tallest student and the stack and the student's desk. Finally, if the stack was not as tall as the tallest student in the class, the student is prompted to estimate how many more books it would take for the stack to be the same height as the tallest student. This task requires that students define essential variables and to make assumptions. The most essential variables are the dimensions of the typical mathematics book in the classroom and the height of the tallest student. They also need to assume that the books are at least approximately the same size and it may even be necessary to assume that the books are stacked on their backs or fronts rather than being carefully stacked from the top edge to the bottom edge.

Birthday Cupcakes is another task intended for primary school that was developed by BAMP. In this task, the students are told that a father, Mr. Ramon, is buying cupcakes for his son's birthday at school. The students are told that the cupcakes come in boxes of 6 and each box contains either all vanilla or all chocolate cupcakes. The students must determine the minimum number of boxes that Mr. Ramon must buy in order to ensure all 27 students in the class each get one cupcake. In this task, students must identify essential variables when they determine that partial boxes cannot be purchased, there should be less than a full box of extra cupcakes left over, and that flavor doesn't matter (mathematically). This task notably requires students to build a model (do the math) on their own, without guidance for the type of model that must be built. Very young students may wish to create a visual model showing boxes with 6 circles representing cupcakes inside, perhaps assigning a number from 1 to 27 to the contents of those boxes. Some students might make use of repeated addition to determine how many times they need to add 6 to the total before they surpass 27. More advanced primary students may make use of multiplication facts in order to determine that 4 boxes will leave 3 students without cupcakes but 5 boxes will be sufficient. The task then requires the student to determine how many cupcakes will be left over. An extension to the task asks students to determine how many boxes Mr. Ramon must buy if each student is to receive one vanilla and one chocolate cupcake. The student must recognize that the model must be refined in order to distinguish between chocolate and vanilla boxes. This can be achieved by drawing boxes with two different colored or shaped representations of cupcakes inside resulting in 10 boxes, 5 with 6 yellow circles drawn inside and 5 with 6 brown triangles inside, for instance. Repeated addition models might make use of two separate columns, both with the exact same work within. Students who have created a multiplication model and have mastered multiplication may simply multiply the previous solution by 2, recognizing that the solution will simply be doubled given that the number of cupcakes per student has doubled.

MATHEMATICAL MODELING HANDBOOK

A Teachers College contribution called Mathematical Modeling Handbook was created in response to the release of the CCSSM. It contains about two dozen modeling activities with student worksheets and teacher material. Each activity brings students through the process of modeling, starting with providing a modeling situation and a “leading question” for the students to answer with the model.

One activity from the handbook is called “Choosing a College.” The leading question for this activity is “how can you choose a college that is most suitable for you?” This question leaves the possible mathematical route very open as it does not instruct, recommend, or insinuate that a particular method must be used. The first question to be answered instructs the student to determine the most important criteria, in their opinion, for what would make a good college. Some suggestions are given for students who might be struggling in determining what important criteria are to them (the suggestions are “athletics, academics, costs, financial aid available, and location”). This is the step in which essential variables are defined. Once the student has determined what matters most to them, they are asked to rank those criteria and to determine a way to indicate how well their choice schools meet those criteria. Here, the criteria and college options are mathematized. The student is then required to use that information to create a model for choosing the best college. At this point, the student is asked to determine if the model helps them make a decision and if it gives the expected results. Some suggestions for some things to consider are given, for example, if the model could be applied to anyone trying to choose a college and if the model helps break ties. This is the model analysis step. The next phase of the activity introduces the concept of decision matrices and leads students through the model revision process. The last question of the activity asks the student to try to find other applications for the model they have developed (see Problem A.8).

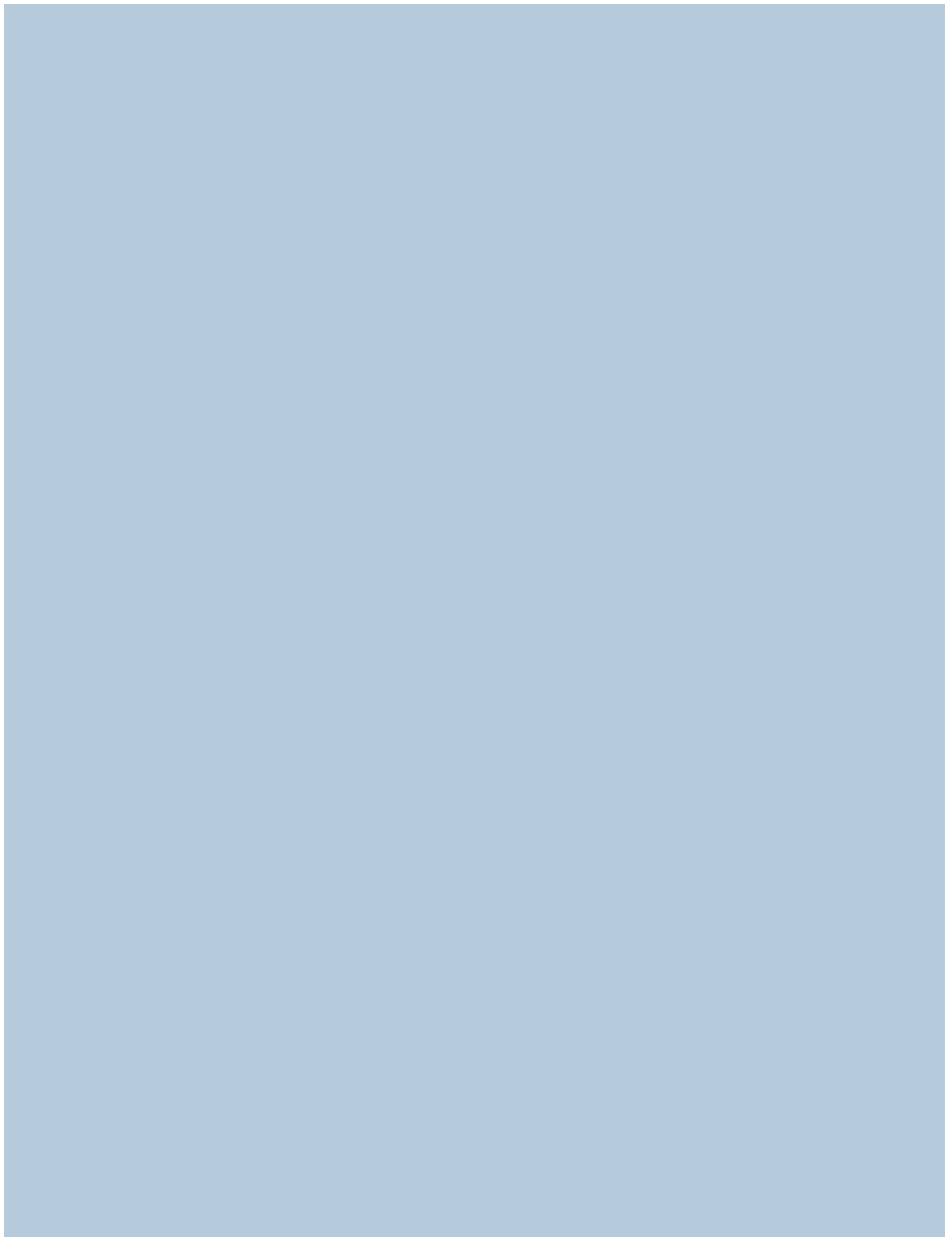
Can other real-life decisions be determined using decision matrices? If so, list them and describe briefly how you would go about creating a model for each.

PROBLEM A.8: MATHEMATICAL MODELING HANDBOOK ACTIVITY, “CHOOSING A COLLEGE,”

FINAL QUESTION REQUIRING MODEL EXTENSION.

FURTHER RESOURCES

While many modeling resources exist, it can be difficult to determine how well each of these addresses the mathematical modeling process and how they can be implemented in the classroom. The Consortium for Mathematics and Its Applications (COMAP) runs a repository of mathematical modeling resources and tasks available. Discussions are underway with SIAM to expand these resources and to encourage members of the mathematics and mathematics education community to upload new modeling tasks. The repository can be found at mathmodels.org.



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APPENDIX B: MODELING EXAMPLES, ELEMENTARY AND MIDDLE GRADES

THE LUNCH BOX PROBLEM

The context for the lunch box problem is answering the question “what should go in a lunch?” We look at this problem through two lenses: the different stages of the modeling cycle and how it might be posed at different grade levels.

STAGE 1: IDENTIFY AND SPECIFY THE PROBLEM TO BE SOLVED — TELLING THE STORY

To engage students in the lunchbox problem, it can be posed as a story where an explanation, decision or strategy is required. If the story is open-ended enough, it encourages the modelers to ask questions that will help them decide how to approach a model and solution.

In prekindergarten–2, teachers might explain to the students that they have to prepare a lunch to take to school in a lunchbox.

In 3–5, teachers might also explain to the students that they have to prepare a lunch to take to school in a lunchbox. The story can be made more complex by adding additional considerations such as size of the lunch box, budget constraints, and nutritional value. For these grades a single additional constraint is probably enough specificity.

In 6–8, teachers might also explain to the students that they have to prepare a lunch to take to school in a lunchbox. The story can be made more complex by adding additional considerations such as size of the lunch box, budget constraints, and nutritional value. For these grades, students may be able to handle multiple constraints.

STAGE 2: MAKE ASSUMPTIONS AND DEFINE ESSENTIAL VARIABLES — MAKING DECISIONS

Once students (perhaps with facilitation from their teacher) have decided what specific problem to approach, they can think about what information they will need to solve the

problem. In simple terms, they need to figure out “what matters?”

In prekindergarten–2, teachers can ask students to say what matters to them without formally noting that the answers could lead to assumptions in modeling. In the case of the lunch box problem, this might be deciding what foods will go into the lunchbox and how much of each food would be appropriate.

In 3–5, in addition to generating individual assumptions about what should go in the lunch, students may consider pairs of assumptions and discuss the compatibility (or incompatibility) of their choices. For example, maybe they want foods with certain vitamins, but they also want foods that the students prefer. In these grades, we would expect estimates and measurements to be more precise.

In 6–8 students can be encouraged to move beyond simple comparisons of assumptions to discussing their implications in the context of the problem. They could also justify why some quantities, such as the number of sandwiches, should be fixed while others should vary. If some information required for the model is not available, students can estimate values for quantities and justify those assumptions.

STAGE 3: USE MATHEMATICS TO GET A SOLUTION – MATHEMATIZING

An important characteristic of mathematical modeling is that students not only make choices about how to narrow the focus of a problem but also are empowered to choose a mathematical approach. The “mathematizing” of the problem is the critical step in reaching a solution.

In prekindergarten–2, students working on the lunch box problem might vote on their favorite menu items. Asked which foods are the most popular, students, as a class, could count the number of votes for each food and compare the numbers of votes. This comparison might be a visual representation or a numeric representation, both of which might help with decision-making.

In 3–5, students might do as above, but begin working in small groups or on their own to make decisions. The representation of the comparison at the end should be a more formal descriptive statistic such as a bar graph or line graph. Students might come up with a simple model like

$$\text{lunch} = \text{fruit} + \text{vegetable} + \text{protein} + \text{carbohydrate} + \text{water}.$$

Then the teacher might work with the students on the units in their equation. For example, the teacher could ask whether the equation has to do with space

$$\text{lunchbox space} = \text{space for fruit} + \text{space for vegetable} + \dots$$

or alternatively

$$\text{number of lunch items} = \text{number of fruits} + \text{number of vegetables} + \dots$$

In 6–8, students might use more sophisticated techniques, like designing a survey, to collect data about favorite items. Students should then work in small groups or on their own to make decisions. The representation of the comparison at the end should be an even more formal descriptive statistic. Students at this level can also work with the additional constraints and if they have used inequalities could add these constraints to the model:

number of lunch items = number of fruits + number of vegetables +
 $1 \leq \text{number of fruits} \leq 3$
 $1 \leq \text{number of vegetables} \leq 2$
etc.

STAGE 4: ANALYZE AND ASSESS THE MODEL AND SOLUTION – DOES IT MAKE SENSE

Because all models involve assumptions and approximations, a critical aspect of mathematical modeling involves assessing the ability of the solution to satisfy the problem. In what contexts does the model provide valuable information? How precise are the solutions? How much do the solutions change if you slightly vary the assumptions, model, or parameters? For example, in the lunch box problem, if one of the length of a lunch box as one the parameters is assumed to be 10 inches, you could ask what might change if the lunch box used was longer or shorter.

In prekindergarten–2, students might represent different food items with blocks etc. in order to investigate how to pack lunches in different size lunch boxes. They might then compare which, and how many items fit into each box. All of this capitalizes on young children’s ability to act out a situation, count, compare size, and make 1-1 correspondences

In 3–5, students might begin to consider if the model they have for the lunch box really make sense. As students’ number sense is becoming more sophisticated at this level (place value and fractions) along with their understanding of magnitude, students can determine if their measurements (actual box) and if their choices for amounts of food really make sense. What are all of the possible values for their model? Do negative numbers make sense? How about fractions or decimals?

In 6–8, teachers can ask students to discuss the pros and cons of their choice for the size of the lunch box, their choices for the amount of food, and the nutritional value of the food choices they made. They can think about the domain of the variables in their models and the range of possible outputs. They can think about how precise the answer needs to be in order to be useful. They can also use graphs to make more sophisticated visualizations of their models.

STAGE 5: ITERATE AS NEEDED TO REFINE AND EXTEND THE MODEL — REPEAT

Based on the analysis in stage 4, a decision may emerge to make some modifications to the model, to the assumptions and or the mathematizing process. Models are never perfect because they are simplifications of reality and so this iterative process might occur more than once.

At all levels, the modeling process is a continual evaluation of the limitations and sources of error in a model based on available information. At all levels, the physical lunch box might get changed to a different size or even become a brown bag. Also, the food choices might change. Ideally, students at all levels will have the opportunity to revise and improve their models based on new discoveries made during the analysis component. Sometimes iteration cannot happen due to time constraints, but even a discussion about possible improvements conveys to students that models are rarely static, perfect solutions to a problem.

STAGE 6: IMPLEMENT THE MODEL AND REPORT RESULTS — TELL THE STORY

Modelers could iterate and improve indefinitely, but deadlines happen. In school and on the job, at some point it is time to declare victory. The good news is that when students report their results, they can explain what they would do to improve their model and in what situations it will and will not apply. This final stage of the modeling cycle provides an outstanding opportunity for mathematical communication.

In prekindergarten–2, since much of the work may be done as a class, the final report here might simply be creating a physical model of the lunch box using manipulatives to represent the box and the food.

In 3–5, the communication skills of the students are evolving and so students can include written assumptions with brief explanations of those assumptions. At this level, students should make basic charts and graphs to support their decisions. Students might also be asked to build a physical model of the lunch box and its context, but with more precision than was seen in prekindergarten–2. Students can make presentations of their models to one another or to another class.

In 6–8, the communication skills of the students are becoming more sophisticated and so students can include a formal written report with assumptions with explanations of those assumptions. The report should also include a description of the mathematizing process for the model. At this level, students should include a variety of more sophisticated charts and graphs to support their decisions. Students might also be asked to build a physical model, to scale, of the lunch box and its contents. Finally, if time permits, students should be asked to do presentations of their models to one another.

BUSES AND PIZZA AND QUOTIENTS

Real world constraints or considerations require problem solvers to continue to think through a solution, or even consider multiple solutions, prior to identifying a final answer. Investigating and solving problems that are mathematically focused, yet necessitate awareness of context, is an early step in the development of mathematical modeling. Below we highlight the approachability and flexibility of a few standards-based problems that take little time to pose, but encourage thought beyond the initial calculation.

One example of a problem in which the context is important and a decision needs to be made is:

THE BUS PROBLEM

An army bus holds 36 soldiers. If 1,128 soldiers are being bussed to their training site, how many busses are needed?¹

In this case, the result of this standard 4th grade calculation (4.NBT.6 in CCSSM), $1128 \div 36 = 31 \frac{1}{3}$, needs to be interpreted in context. The extent to which the problem has a single agreed-upon answer depends first on the contextual need for the number of buses to be a whole number. The reason for why the answer may be 32 rather than, say, 33 or 40, depends on the implied practice of using the least number of buses possible. What if other factors need to be considered? For example, assume that soldiers need to travel with their group, or “platoon” and that platoons can vary in size from 26 to 32 individuals. How many busses are needed now?

To solve this problem you can encourage students to work in groups to find a solution associated with platoon size and ask them the following questions:

- Is it possible that all platoons are the same size? Why or why not?
- If not, what combinations of platoon sizes are possible?
- How many busses are needed to transport the soldiers using your platoon system?

More than one correct answer is now available. When students need to consider the results of calculations in context, there are opportunities for students to reflect on the assumptions that are made and the credibility of the calculations they choose to do.

LET’S CONSIDER THE PROBLEM $34 \div 8 = ?$ ²

This problem — actually a consideration of five problems, listed below — highlights how context is needed in order to determine an answer to a seemingly straightforward problem. In particular, the answer to $34 \div 8$ is not trivial. It could be 4 remainder 2, $4 \frac{2}{8}$, $4 \frac{1}{4}$, 4.25, 5, or perhaps in some cases “it can’t be done.” Students in the 3–5 grade band can be asked to consider the following five problems, all requiring the calculation of $34 \div 8$, and identify a solution that makes sense.

ROSA AND THE TOMATOES

One of Rosa’s great pleasures is growing tomatoes. She shares each batch equally among her eight friends and keeps any leftovers for herself. One week Rosa picked 34 tomatoes. How many did each of her friends get and how many did she get to keep?

THE RELAY RACE

A group of friends wants to run in a 34-mile relay race from Originville to Numberline, a one-way route that follows a straight road. Each runner is able to complete 8 miles individually. How many runners does the team need to be able to complete the relay?

A VEGETARIAN LUNCH

Yvonne has a favorite vegetarian sandwich at a local café. The sandwich costs \$8. Yvonne has budgeted \$34 for lunches this week. What is the greatest number of days she can have her favorite vegetarian sandwich for lunch?

BAKING COOKIES

Dave is baking cookies. One batch yields 34 cookies. Dave wants to share his cookies with his 7 friends and himself. If he shares the cookies equally among the 8 people, and cookies can be split into smaller pieces, how many cookies will each person get?

BIRTHDAY PARTY SEATING

Judith invites 33 people to her birthday party at a fancy restaurant. The restaurant has tables that seat 8 people. How many tables will she need to reserve?

The first four questions are intentionally prescriptive; thus when investigated collectively they emphasize the importance of perspective in problem solving. Answers to the first four questions, are:

- Rosa: 4 remainder 2. Friends get 4 tomatoes each and Rosa gets to keep 2 for herself.
- Relay Race: 5 runners are needed.
- Lunch: Yvonne has enough money budgeted to have her favorite sandwich for 4 days.
- Cookies: $4\frac{1}{4}$ or $4\frac{2}{8}$.

Investigation of the fifth question allows students flexibility to engage in mathematical modeling based on their experiences. For example, one student may note that you need to “round up” to five tables in order to accommodate all the students comfortably. While another student may remark that ideally you want to keep the group together and could advocate for “rounding down” to four tables, noting that an extra chair could be pulled up to two or the four tables.

Students in the prekindergarten–2 grade band can use their developing number skills to investigate $34 \div 8 = ?$ (or similar problems) in context as well. For example, Baking Cookies

can be presented directly to students in this grade band; ideally with real (or paper stand-ins) present. Have the students investigate ways to share 34 cookies with eight friends. Can they find the fairest way to share?

Slightly altered phrasing can connect the problems to familiar experiences as well. Rosa and the Tomatoes can instead become

ROSA AND THE PARTY FAVORS

It's almost Rosa's birthday. She has 34 bouncy balls that she will hand out as party favors to her eight friends. She will hand out the balls equally and gets to keep any leftovers for herself.

To encourage additional creativity in using number sense as a problem solving tool. Consider organizing students into groups of eight and giving each student 34 of a particular item such as markers, small "bouncy balls", beads, or cookies and allow them to identify a way to share the items with the other classmates in the group. In some cases, the students will need to make difficult decisions. It will be important for them to explain how and why they think their solution is fair.

In grades 6–8, students can use their developed arithmetical skills to start solving problems similar to the Bus Problem and $34 \div 8$. Current grade band mathematics, such as descriptive statistics, can be incorporated to produce a more sophisticated solution. Consider the following open ended problem in a class of 20 students.

PIZZA PARTY

Your class is having a pizza party. How many pizzas should you order?

An initial solution can be determined by making the following assumptions:

- All guests attend.
- Everyone likes the same type of pizza
- Everyone eats two slices
- There are eight slices in a pizza.

The answer is $(20 \cdot 2) / 8 = 5$ pizzas. But are these assumptions reasonable? In this case students can develop a survey to learn more about the pizza eating preferences of their classmates. Subsequently, the data can be analyzed using statistical methods in order to develop a solution that determines how much and what types of pizzas need to be ordered.

**TRAPEZOID
TEATIME**

In some instances, modeling tasks ask students to engage more intensely with the real world context and require more connections between the mathematics and the real world context. Such tasks, which appear in mathematics teacher journals as grade specific, might appeal to teachers of other levels. To make such tasks usable, modifications to the presented task would be necessary to make the task appropriate for different grade levels. In this example, we identify how modifications may be made to the original task to make it more appropriate for the content of specific grade levels. A common example is the Trapezoid Teatime task adapted from <http://yspmcps.wikispaces.com/file/view/Trapezoid+Teatime-Teacher+Packet.pdf>.

In the original form, this task would likely appear in a grade 5 or 6 class. But, for a moment we explore how this problem might be modified for other grade levels.

In prekindergarten–2, students might be asked to explore options for new tables in their school cafeteria. In doing this, students would be asked to determine how many students could sit around tables of different shapes and then display their findings. Students might also be asked to put tables of same and different shapes together to yield different shaped tables and again estimate how many students can fit around the table.

In 3–5, in addition to the task listed in prekindergarten–2, students might be asked to explore options for new tables in their school cafeteria. In doing this, students would be asked to more precisely determine how many students can sit around tables of different shapes, how the tables might be arranged in the cafeteria, including using the Cartesian plane, and how many students can sit in the cafeteria at one time with the arrangement. Again, students might be asked to data displays to show their finding. Students might even be asked to explore putting same and different shaped tables together to determine how many students might fit in the cafeteria.

In 6–8, in addition to the task listed in prekindergarten–2, students might also be asked to explore options for new tables in their school cafeteria. In doing this, students would be asked to determine how many students can sit around tables of different shapes based on actual size of tables and chairs. Algebraic might be used to determine the number of students that can be seated around tables if multiple tables are put together to form larger tables. Students might be asked to create a layout, to scale of the table arrangement, and examine how rigid transformation can be used to rearrange the tables to increase or decrease the capacity of the cafeteria. Students would also include a formal budget for purchasing the tables and chairs along with a data display of their findings.

It should be noted that tasks such as this, with modifications, have been used successfully used at the grade levels mentioned as part of the IMMERSION project.

The Trapezoid Teatime problem is one that might have an appeal to many teachers, but

the task may only be grade level appropriate with modifications. This narrative provided examples of how these modifications might be made to a task to make it more grade level appropriate.

BOTTLE GAME ACTIVITY

The focus of the IMMERSION program⁹ is to understand how elementary school teachers' views of mathematics and teaching change as a result of engaging their students in mathematical modeling. After professional development, teacher study groups develop and implement modeling lessons in their classrooms. This task was developed as one of those lessons.

Several of the problems developed by IMMERSION teachers have had a design focus. For example, students might develop a game or make plans for a trip or design a playground. These design problems intersect with problem-based learning and 21st century skills and easily engage the students in creative activity. However, in the context of mathematical modeling, a problem can arise if the work focuses too long on the design and development phase, without clear connections to mathematics. In these cases, instead of providing a rich foundation of the task, the mathematics arises as incidental or summative (used to report on what was designed) rather than as critical and formative (used to make quantitatively justifiable choices).

The Bottle Game task was developed and tested by a group of teachers in the Pomona Unified School District (PUSD) in Pomona California in 2015. It was first piloted by teacher leaders Grace Greenleaf and Sabrina Ortega then further developed and tested by teachers Yvette Harris, Mireya Jimenez, Nicki Lew, Jamie Santana, and Joseph Shim working with Rachel Levy from Harvey Mudd College.

This version of the problem was developed after testing in 3rd, 5th and 6th grade classes. Days 1, 2 and 3 represent an example of how to facilitate the modeling problem. Then there are some alternative or extension questions.

DAY 1

Teacher launches the general problem:

- Suppose you are designing a game with 5 water bottles and 5 bean bags. What questions would you need to answer in order to design the game?
(5 min)
- Teacher divides students into teams of about 5.
- Teacher provides a brainstorming task: Record all suggestions without judging them. If you want to add an idea, say “yes, and...” rather than “no” or “but.” (This technique helps students generate more ideas.) Each team records ideas (paper or computer).
(10 min)

- Teacher regroups class and collects ideas. They may include suggestions like:
 - > How many bottles should be used?
 - > How should the bottles be arranged?
 - > How far away should the player stand?
 - > Should all players stand the same distance away?
 - > How many tries should the player get?
 - > How should points or prizes be awarded?
 - > (15 min)

- Teacher selects one of the questions for the groups to tackle first: We'll start with the question "how far away should the player stand if the game has one bottle and one bean bag?" Discuss with your group how you could collect, analyze and display data to answer this question and justify your conclusion. If the teacher would like to make sure the students use a familiar concept such as proportion, mean/median/mode or a way of displaying data such as a table or histogram, the teacher can suggest these tools at this point.
(5 min)

- Student groups discuss approaches.
(10 minutes)

- Teacher regroups class to hear some ideas. Together they go over the plan for day 2. The group use the remaining time to plan what they will do on Day 2.
(20 min)

DAY 2

- Teacher reminds the students of the problem and plan for the day. Teacher answers any questions about the process.
(5 min)
- Groups have a little time to regroup and finish planning
(10 min)
- Groups collect data. Teacher could decide that the data should be collected as a proportion (success/tries) or a table, depending on the target mathematical concepts.
(20 min)
- Students create a poster. The teacher or students assign roles so that each group member has a clear task, such as
 - > Design, intro: Create the overall design of the poster including section titles and an introduction explaining the goal of the data collection.
 - > Data collection: create a written description explaining how they collected data and justifying choices they made.
 - > Figure: create a diagram showing what they did.

- › Table: Create a table with the quantities they measured, how many times they repeated the data collection, and appropriate units (time, length).
 - › Conclusion: explain and justify their answer to the question. It might be helpful to have a draft and then a revision.
- (30 min)

- Teacher reminds students that they will present their posters the next day.

DAY 3

- Teams practice explaining their posters from Day 2. Each team will have 5 minutes to present their poster, with each member of the team presenting their part for one minute. They practice with timers.
(15 minutes)
- Teams present their posters to the class. Alternatively they can record videos of themselves presenting their posters.
(50 - 60 minutes)

EXTENSION ACTIVITY I: BUDGETING

Teachers who want students to practice working with decimals, money or simple algebraic expressions could do so through a budgeting problem.

Version 1 A game manufacturer would like your team to create a carnival game with 5 water bottles and 5 beanbags. Water bottles cost \$1.00 each and beanbags cost \$1.25 each. The game will be played at a carnival. The company expects 175 children to play. Small prizes cost \$0.50 each, medium prizes cost \$1.00 each and large prizes cost \$3.25 each. The company plans to charge \$250 for the game (including the prizes) and they want \$100 profit, so you have \$150 to spend on each game and the prizes. Plan how to spend your budget and use mathematics to show that your plan for the game will work for 175 children. Make a poster with:

- a drawing of the game
- the rules of the game
- Extension: If you had to revise your budget to \$100, what would you change? Justify your answer.

Version 2 Do the same problem as above, but have students create variables for the number of each item and write an algebraic expression for the total cost. Then have them plug in the numbers they will assign to each variable and show that they have met their budget.

Version 3 Have students use a spreadsheet to do the budget calculation. Make sure the students are able to enter any formulas that they use into the spreadsheet themselves, since that will help them demonstrate their mathematical understanding.

Version 4 Adjust the difficulty of the problem by changing the number of children, the budget and the costs associated with each item. The costs could be whole dollars, familiar amounts, such as multiples of \$0.25 or less common amounts such as \$1.31.

EXTENSION ACTIVITY 2: STUDENT-DESIGNED QUESTIONS AND APPROACHES

After completing the original three-day activity, teachers may choose to revisit the bottle game using a new mathematical concept. For example, maybe students are working on:

- a way of displaying data, such as charts or histograms,
- plotting points on the Cartesian plane,
- computing measures of central tendency such as mean/median/mode, or
- measuring in metric units.

Here are some examples of questions students might ask about the bottle game that they could answer using some of the skills above. For each question there are several ways to go about answering them — this set of Q&A is just to provide for each question an example of the direction students might go. Together the questions and answers illustrate the many possibilities for modeling problems and solutions given the context of a simple game. Above, students came up with the questions. Here, the idea is for students to come up with the solution approach.

Question How far do people in our class throw a beanbag?

Approach Have each student throw three times. Record the median (middle) distance as the score for each student. The class could discuss whether to use the maximum of all the scores (this is the farthest the class can throw), the minimum of the scores (this is how far everyone can throw) the mean or median of the scores (this is how far we throw on average) or the mode (this is how far most people throw).

Approach Children in K-2 could throw the bean bag, measure how far they threw then add their number to a histogram using a sticker or other marker.

Question How many throws does a person need to make before we have a good sense of how far they throw?

Approach Have each person throw 20 times. See how the mean changes as they throw more times. Students could display their data in a table or plot the number of throws versus the mean.

Question How much do people's throws vary?

Approach Have everyone throw 5 times and find the difference between their longest and shortest throw. Plot all the differences on a histogram. Discuss the result.

Question How far should players stand from the bottle when they throw the beanbag?

Approach Have each person stand 1 meter away and try 3 times to hit the bottle. If they

miss the bottle, they record that distance and stop. If they hit it, they back one meter and try again.

Question Is throwing distance related to the person’s height?

Approach Have everyone record their height and how far they throw (they could take an average of three throws). Then they can plot height versus throw distance. They could discuss which plots indicate that there is a relationships and which ones don’t. They could also discuss what kind of plots might make it difficult to be certain of the answer the question and which ones seem to show a clear relationship (correlation).

Question How many bottles should the game have?

Approach Pick a distance to stand. Mark on the ground where you will put 1, 2, 3, 4 or 5 bottles. Give each player 3 tries with each number of bottles. Use that data to make an argument about what the “best” number of bottles is. They could take into consideration how they want to define a game that is “too easy,” “too hard,” “just right” or even how they want to define a game that is “fun” or “fair.” The teams of 4 students could each collect data for their team, then the whole class could combine the data, then each team could interpret the data and make an argument according to their own assumptions about what makes the game work best.

THE FOOTPRINT PROBLEM

A traditional set of data collection and modeling problems encourage students to explore proportional reasoning by having them comparing different body lengths (such as height to armspan, foot length to hand length, arm length to leg length, etc. A version of this type of exercise was published as a version of the Footprint Problem by Kara Imm and Meredith Lorber.⁴ To illustrate how mathematical modeling unfolds in a P-8 classroom, we will consider how Imm and Lorber engaged Lorber’s sixth grade students this Footprint Problem as a modeling activity and relate it to the modeling facilitation cycle in the elementary grades chapter (see Figure 2.3 from Chapter 2). The keywords from this figure will appear in italics.

We begin with a portion of Imm and Lorber’s account of the experience (quotes block indented). They talk first about *developing and anticipating* the task by selecting and modifying it for their students.

We started by looking for a good modeling task and modified the Footprint problem with Lorber’s students in mind. With students gathered together at the front of the room, I [Kara] shared the story of a recent school visit (see Problem B.1).

Once the story had been presented and copies of the footprint were passed around, we paused to invite students into the context. “I don’t know about you,” we noted, “but

seeing this footprint raised a lot of questions for us. We are going to stop and ask you all to write down any questions you have about the person wearing this shoe or perhaps about the situation itself. What do you want to know? What does this make you wonder about?" (p. 49)

I've been traveling to a bunch of schools, and I want to share with you something that I witnessed recently. Some of you know that we have some very large high schools, with two, three, even four thousand students in the same building. So, when I arrived at one of these schools the last few times, several adults and kids were talking about an alleged vandal who had been sneaking in at night after the security guards had gone home. Can someone remind us what a vandal is? And what does it mean to vandalize something?

So, we don't know much about this suspected vandal except what he or she left behind (messed up lockers, a broken-into chemistry lab, a torn up bulletin board). However, recently the vandal must have stepped in something on the way to or from the scene because this person left a footprint—a shoe print, more precisely. Now with digital re-creation technology from the Department of Education, school security obtained an image of the shoe they believe this person was wearing. Because I happened to be visiting a teacher that day, I actually have a copy. Would you like to see it?

PROBLEM B.1: LORBER'S STUDENTS HEARD INFORMATION ABOUT A REAL-LIFE EXPERIENCE.

Asking the questions, "What do you notice?" and "What do you wonder?" can be very useful at many components of the modeling process. Here it is used to launch a question. In our modeling facilitation cycle (Figure 2.3) this is the *organize* part of *enacting*. The teacher also could ask students to talk about what they notice and wonder as a general framework when they comment on each other's work during the modeling process. In some cases, it may be valuable to provide additional guidance by prompting discussion about the one clue — the shoe. As a result "What do you wonder?" may be supplemented by "What is important about the footprint?" or "How can we use the footprint to help solve the mystery?" and taking care to let students explore and to keep the investigation authentic. While students are learning how to describe their world in quantitative terms it may be appropriate to introduce (or further emphasize) measurement using this problem as a basis.

As a class, we decided to investigate the question "How tall is the vandal?" The question, as is, was not noticeably different from other school math problems. So we extended the question to ask, "Is it possible to determine the height of any person, based only on their shoe print? If so, how would you do it?" This subtle extension of the mathematics moved the task from mere problem solving ("How tall was the person wearing this shoe?") toward modeling ("Does a model exist for predicting height from any shoe?"). (p.49)

Note that the teachers launched a very open-ended problem, collected many ideas, then helped the class *pose questions* and come to consensus about a direction of inquiry. The openness of the problem could also have been emphasized by asking for the person's approximate height. This means the students in the class posed the same mathematical problem, but might make some different assumptions and use different solution strategies. By anticipating different mathematical approaches, the teacher can guide students to focus on an aspect of the problem that both interests them and provides opportunities to use mathematical tools (either those previously learned or those currently under study). Teachers can *monitor* teams to see if they choose a strategy that motivates a new mathematical topic. Teachers can also revisit an old modeling problem after the students have acquired a new tool.

Next, we asked students to generate a list of items that they would need to investigate this situation. We compiled a class list and then reminded students of the tool table that was permanently located in one corner of Lorber's room. ... Because tool choice and strategies are often linked, we witnessed several examples of students having to question, resolve, and negotiate these choices in their groups. Would they measure in inches or in centimeters? Could strips of large graph paper help them average the data? Would measuring the width of the shoe or maybe the perimeter be helpful? Could self-reported heights be trusted, or should students verify these claims? (p.51)

Here you see the students *build solutions* and the teacher *regroup* the students and facilitate the process of making assumptions and defining variables. They are also monitoring the students as they move toward mathematical strategies.

Students wrestled with the idea about what it means to form a relationship between two quantities and how to represent and express that relationship. One group imagined a person's height as measured in shoes (see Figure B.1), another student argued that the ratio between her shoe and the suspect's shoe gave insight into the ratio between their heights (see Figure B.2), and several tried to distill the mathematical relationship between shoe length and height using a ratio table (see Figure B.3).

Words such as negotiate and wrestle call attention to what many people know as "productive struggle"^{5,6} The teacher monitors these activities and notes what approaches students are taking, what mathematics they are employing, and whether there are ideas emerging that need more "mathematization." For example, if a group claims something is "best" or "fastest" the class may need to regroup and discuss how they can use mathematics to quantify and measure how something is "best," so that they can justify those claims using mathematics. Or perhaps the group is actually seeking a solution that is feasible, rather than optimal, so with young students you might ask whether it is the "best solution" or just a "solution that works." Then students could work with the teacher to define what it will mean

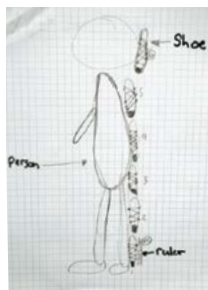


FIGURE B.1: A PERSON'S HEIGHT MEASURED IN SHOES

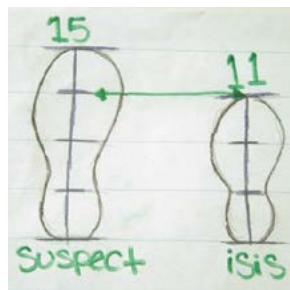


FIGURE B.2: A COMPARISON

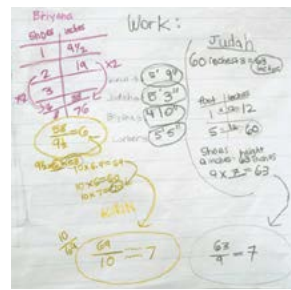


FIGURE B.3: THE SUSPECT'S SHOE LENGTH AND HEIGHT IN A RATIO TABLE

for a solution to be useful or to work. At this point the students are “doing the math.”

Meanwhile, others thought of the problem additively, noting the difference between the vandal’s shoe length and their own, and did not see the problem multiplicatively at all. We did not correct this misconception because Lorber’s yearlong plan included returning to proportional reasoning a little later. Additionally, Lorber’s classroom and school culture ensured that students would receive peer feedback; whole-group feedback using a protocol; and, for those who continued working on the task, feedback from community members during a school-wide event. Only when additive reasoning continued to fail as an approach would students come to consider a different strategy and revise their thinking. Students neither followed the same paths nor arrived at the same models. However, we witnessed the ways in which each student developed some new mathematics. Some saw a ratio table for the first time and began to try it out in other contexts. Others wrestled with ideas related to measurement and data collection. (p. 52-53)

Here the teachers make an interesting point about misconceptions. Teachers must make choices about how to spend class time, and when it would be an appropriate time to address a particular misconception. In some ways this is like addressing student writing, in which it might not be appropriate to correct every error and instead respond to the big ideas such as organization or an aspect of mechanics such as apostrophes that is a current area of study for that student or for the class.

At the end of the modeling session, students debated the accuracy of their solutions and tried to *validate their conclusions*. For example, the teachers note that “We even witnessed a contentious debate around precision: whether rounding data before or after they are averaged will make a significant difference.” and “A class conversation about whether or not a 7 foot 9 inch woman really existed, for example, allowed students to interpret their mathematical results in the context of the situation not depend on their teacher to verify

their results. They also noted times that the modeling process motivated the use of new mathematics, such as the ratio table, measurement and data collection. These ideas could be revisited in a lesson more focused on the mathematical tool, or could be used by students in another modeling problem.

**REVISITING THE
FOOTPRINT
PROBLEM**

In their article, Kara Imm and Meredith Lorber talk us through their enactment of a mathematical modeling activity with grade 6 students. We can *revisit* the footprint problem with different mathematical targets or grade levels in mind. As the activity is launched by Imm and Lorber, the students use mathematical modeling to predict the height of the person who made the footprint. Their students take a number of solution approaches, including using non-standard units of measure and proportional reasoning. Another approach could be to plot foot length versus height of everyone in the class and then predict the height of the suspect by seeing where the mystery footprint fits in the data trend.

These approaches to the footprint problem take a predictive modeling approach. What other kinds of modeling tasks could the footprint problem inspire? The youngest students might focus on measurement, comparison and data/extrapolation. For example, the teacher might use a footprint from one of the students' shoes. Then each student could make a mathematical argument about how that footprint relates to their own footprint. Could it be from their shoe? Why or why not? Another task might be developed by taking a footprint from one of the teachers or administrators at the school. Students could make arguments about whose footprint it is or estimate a range of possible grade levels of the student who made the footprint.

For older students, the teacher could pose the same footprint problem as a completely open task. For example, the police found the footprint at the scene of a crime and they want to know what the students can tell them about the person who left it. If students don't start thinking about height, the teacher could *regroup* the class and provide additional information. For example, perhaps the police have the heights of 60 suspects (from photos). The class could develop approaches that the police could use to decide the most likely heights of the person who made the footprint.

Sometimes a modeling task can lead to others. Thinking more broadly about shoes, students could also engage in problems where they rate and rank shoes in the class.

How many ways could you compare shoes? Which qualities are important in a shoe? How does the importance of one quality relate to others? Younger students could classify shoes by color and create a chart to identify the most popular choices in the class. Older students may identify and rate important qualities such as color (5 = love it, 1 = hate it), fit (5 = great, 1 = terrible), and use (5 = what I need for the occasion 1 = doesn't work for this occasion),

want (5 = really want, 1 = don't want). When you buy a shoe, maybe fit and use are 3 times more important than color and want. So now for any set of shoes, someone could score them and a company could see which ones people would buy. These types of rating and ranking decisions can be visualized in a chart or matrix.

Students could also think about the geometry of the footprint. They could try to figure out what shapes you would combine to make a footprint (2D) or a shoe (3D). What does the shape formed by the footprint tell you about the movements or weight of the person wearing the shoe? They could look at tread patterns and think about the shoes in their class. Which aspects of the shoes are there for looks and which ones have a function? How could they categorize and evaluate shoes based on these attributes?

More extensions of the footprint problem could be explored if you “find” a set of consecutive footprints. Was the person who made the footprints moving in a certain direction? How do you know? How fast (or slow) was the individual moving? Younger students might investigate these questions by measuring step lengths they create by walking, hopping or running. What happens if a teacher creates a set of footprints too? Older students working on the open version of the problem could use this additional information to learn more about the suspect.

A good modeling problem is able to engage students across grade levels with different mathematical backgrounds and is accessible from multiple mathematical perspectives. The footprint problem provides us with an example setting from which we can explore the many mathematical approaches one can use to better understand an authentic real-life situation.

THE PLAYGROUND PROBLEM

One of the perceived challenges of mathematical modeling is “fitting it in” the existing curriculum because the curriculum is driven at each grade level by specific content and processes that all students are expected to know and understand. The modeling process lends itself toward rich learning in many domains and addresses many of the mathematical practices, including MP 4 of the CCSSM. To illustrate how one particular modeling task can be modified to address different grade level mathematics content expectations, we consider the school playground problem, chosen because the context will be familiar to students. It is quite surprising to see the number of Content Standards that can be addressed by one modeling task and so appropriately chosen tasks such as this one can address many Content Standards.

In each of the task descriptions for the different grade levels, there are lists of expectations for the students. After each of those expectations, the different Content Strands that are addressed are listed:

MODELING TASK FOR PREKINDERGARTEN-2: THE PLAYGROUND

You have been put in charge of designing a playground and need to make a presentation to the class about your design. Your tasks for designing the playground are as follows:

- Based on input from your classmates, you should choose six pieces of equipment or spaces to include in the playground; (Number and Operation, Data Analysis and Probability)
- You should create a display that communicates data/information about your choices in sub task 1; (Data Analysis and Probability)
- You should estimate what you think the cost of the equipment might cost; Weigh the options of adding different equipment or taking some pieces away; (Number and Operation)
- You should estimate how much space you might need for your playground; (Geometry)
- You should choose appropriate shapes or a few shapes that you put together to represent the space that might be taken up by your equipment; (Geometry)
- You should lay out the playground on a grid using the input from your classmates making sure to say how you chose where to place the equipment; (Geometry)
- You should determine the number of each type of shape used and sort them according to similar attributes... number of sides, symmetries, etc.; (Number and Operation, Data Analysis and Probability)
- You should write a brief description that describes how your equipment was placed; This description should include appropriate measurements; (Geometry, Measurement, Data Analysis and Probability)
- If you could only choose one new piece of equipment, how will you decide which piece of equipment to use; (Data Analysis and Probability)

MODELING TASK FOR 3-5: THE PLAYGROUND

- You should select and include at least six pieces of equipment (slides, swings, sand boxes, etc.) one of which is a basketball court;
- You should design a survey for classmates about which equipment to purchase; The results should be included in the report and be involved in the purchase and placement of the equipment in the playground; You should include some form of graph or table should be included; (Data Analysis and Probability)
- You should determine what the cost should be for the playground and include comparisons for different equipment; (Number and Operation, Algebra)
- You should use two-dimensional geometric shapes to layout the playground; Larger shapes should be used for larger pieces of equipment while smaller shapes should be used for smaller pieces of equipment; (Geometry)
- You should lay out your playground using the first quadrant of the coordinate plane; You should identify how your playground is laid out using the coordinate plane; You should name the positions of the corners of the shapes used to represent the equipment using with ordered pairs; (Geometry, Measurement)

- You should describe the layout of the playground including distances between the objects; (Geometry, Measurement)
- You should create scaled three-dimensional models of the playground equipment using geometric solids; (Geometry)
- You should include a fence around the playground and incorporate the cost of the fence into the budget; (Number and Operation, Measurement)
- You should minimize the space for the rectangular playground; (Algebra, Geometry, Measurement)
- You should include a description of how the shapes might be rotated and still fit in the same space should be included; (Geometry)
- You should include a written report to accompany your presentation to the class;

MODELING TASK FOR 6–8: THE PLAYGROUND

You have been put in charge of designing a playground and providing a report and a scale to the local elected officials. Here are the expectations:

- You should include at least six pieces of equipment (slides, swings, sand boxes, etc.) in the playground and it should fit into space with a fence on all four sides that is roughly 50 feet by 100 feet; (Measurement)
- You should complete a survey of classmates about which equipment to purchase; The results should be included in the report and be involved in the purchase and placement of the equipment in the playground; An appropriate graph should be included; (Data Analysis and Probability)
- You should include a budget that does not exceed \$100,000 and that includes the cost of constructing a fence on all four sides; (Number and Operation)
- You should include a scale model of the equipment that uses 3 dimensional geometric shapes, perhaps constructed from cubes appropriately fused together to create objects that represent the equipment, should be included; (Number and Operation, Geometry, Measurement)
- You should lay out the objects created in sub task 5 on a coordinate plane so that the area that each will take up can be determined; (Geometry, Measurement)
- You should mention what may happen if the equipment is rotated in some way; (Geometry)
- You should include least two feet for a walkway around all of the equipment; The walkways should be represented on the scale model; The total square footage of the material needed to construct the walkways should be included as well; (Number and Operation, Geometry, Measurement)
- You should construct the model to scale; (Number and Operation, Geometry, Measurement)
- You should include the surface area of the structures as well as the volume (as appropriate); These will be needed to determine the amount of paint needed and the amount of liquid or sand needed to stabilize certain playground equipment; (Measurement);

- You should include a report to accompany the presentation should be included;

We now turn to a discussion of how this problem evolves through the various grade bands while simultaneously increasing the cognitive demand. The layout of the playground starts in prekindergarten–2 with students laying out their playground on a grid and using words to describe the position of the playground equipment. In grades 3–5, more exact positions of the equipment using the coordinate plane. Finally, in grades 6–8, the area for the playground is pre-determined and the layout of the equipment, now done to scale, must fit the constraints. In this progression with the layout of the playground, the increase in cognitive demand can be seen as the task evolves through higher-grade levels. The representation of the equipment evolves from two-dimensional shapes in prekindergarten–2, to three-dimensional representations in grades 3-5, to three-dimensional representations that are to scale in grades 6-8. Similarly, the cost or budget for the equipment evolves from an estimate of the cost of the equipment using whole numbers in prekindergarten–2 to a formal budget with constraints that also includes fences etc. in grades 6–8.

A quick caution should be included with this particular task, particularly when teaching mathematics. Much of this task involves designing a playground and while design skills are important and a key component of 21st Century Skills, the focus must be on the mathematics. The design of the playground can sometimes overshadow the main learning that is to take place. It is true the design skills and mathematics complement one another nicely and design skills provide a solid context for teaching mathematics, but caution should be taken that the design skills do not inhibit the mathematics learning.

We finish this session with some notes about implementation of this project into the curriculum. At each grade level, there is a complete modeling, with multiple sub tasks. In many instances, these sub tasks can be taken as stand alone tasks or combinations of them might be taken together. This decision would be driven by the teachers' needs. For example, the sub task 2 in grades 3–5 would address aspects of data analysis and probability. In some instances, specific content may need addressed. However, more importantly, some of the sub tasks may be used to introduce students to the different stages of the modeling process. For example, one stage of the modeling task is making assumptions. Sub tasks 1 and 2 in grades 6–8 ask students to select equipment based on a survey and that will fit into a certain amount of space.

THE MINI GOLF PROJECT

In this example, we explore a modeling task as an assessment. In some instances, a modeling project may be the ideal assessment (project based or performance) at the completion of a unit because it provides opportunities for students to connect, use and apply many mathematical ideas at the same time. The following example is appropriate for grades 6-8, but has also been used as an assessment in pre-service early childhood, elementary, and middle level teachers' mathematics content courses.

MINI GOLF PROJECT

In this assessment project you are going to be designing two, six hole plus one practice hole (that is a regular polygon), miniature golf courses. The two courses must have the exact same six holes, but moved around to different locations. However, the second design should have different practice hole that is still a regular polygon.

THE FIRST DESIGN

Using GeoGebra or another dynamic geometry program, you should create a six hole miniature golf course plus a practice hole. In your report, identify the coordinates of the corners of the holes with ordered pairs, as well as the location of the actual cup. Also include the area and perimeter of each of the holes. If any of the sides of the holes you design are parallel, they should be made in the same color in the GeoGebra design. In addition, you should use the angle measure feature of the dynamic geometry program feature to identify the measures of the angles that make up the corners of your holes. If your holes can be made by composing polygons together, make sure to indicate how to compose the polygons to construct your holes (i.e. a square plus a triangle).

Two of the holes must include obstacles. The obstacles should be made from combining platonic solids or other known 3 dimensional objects (i.e. a silo from a cylinder and a half sphere). Identify the surface area of the objects and the amount of water or sand needed to fill the objects so that they can be grounded.

For the practice hole, identify the regular polygon that you used as well as the measures of the interior, exterior, and central angles of the polygon.

Your design might also include walkways around all edges of the holes. That walkway must be at least one block (on the grid) wide all the way around. This means that there must be space between all of the holes to accommodate this. Determine the area of the walkways... hint... square counting.

SECOND DESIGN

Your second design should simply use rigid transformation to relocate the original holes to their new positions. The transformations should be included in the report. You should also determine the area needed for the walkways with the new design, if you did that in the original design.

COST IT OUT

Now your challenge is to cost out the construction of each of the golf courses. You need to have wood around the perimeter of each of the holes. Determine the total cost of the wood to build all of the holes. You will need to purchase the carpet to carpet the holes. Determine the cost of the carpet to cover all of the holes. Because of how carpet is purchased, you may need

to purchase extra carpet. You also need to determine the cost to paint each of the obstacles and the cost of the concrete needed to build the walkways, which must have a depth of at least three inches. If you are uncertain about costs... check out an internet site for a major home improvement store.

**TRANSFORMING
FAMILIAR
PROBLEMS INTO
MODELING
PROBLEMS**

Sometimes you can find great modeling problems in the community. For example, maybe a café wants to know how much milk to order each week so that they don't run out and yet the milk doesn't spoil. Other problems come from modeling activities that have been tried in the classroom and published. Here we will show some ways to turn familiar textbook problems into modeling problems:

KINDERGARTEN: COUNTING AND TALLYING

Original problem: Count from 1 to 10.

Modeling problem: Are we eating enough vegetables and fruit at lunch?

Approach: After lunch, have the kids tally each day how many fruits and vegetables they ate.

FIRST GRADE: SUBTRACTION: START UNKNOWN

Original problem: $? - 2 = 5$

Modeling problem: You have a lemonade stand and you want to sell lemonade for \$1 a cup. You have to pay back \$2 for the lemonade mix, but you got the water and cups for free. How many cups of lemonade do you have to sell to be able to keep \$5 at the end of the day? At the end of the day, how much money do you need to have in your money box?

Approach: Maybe they will use subtraction. MONEY IN BOX - \$2 = \$5

SECOND GRADE: COMPARISON

Original problem: which number is bigger (given two integers)

Modeling problem: This month is it getting warmer, colder or staying about the same?

Approaches: Kids could look at a thermometer at the same time each day and record the number on a chart. Then they could use arrows to show that the temperature went up or down that day compared to the last day. The teacher could also use < and > signs if that seems appropriate.

THIRD GRADE: DIVISION

Original problem: If you have 1000 pretzels and the class eats 100 pretzels each day, how many days can the class eat pretzels?

Modeling problem: If we have a bag of pretzels and we want it to last the whole week, how can we figure out how many pretzels to give each kid each day?

Approaches: Maybe the class wants to count all the pretzels, so they need to be passed out in piles to all the students. Maybe they want to put the pretzels in five piles and shift

them around until they count the same number in each pile. Then they could figure out how to divide up the pretzels in one pile among all the students. Finally they could represent their approach with pictures and with symbols.

FOURTH GRADE: AREA OF A RECTANGLE

Original problem: Find the area of a rectangle using Area = length x width.

Modeling problem: Find the surface area of the top of your desk using a piece of graph paper. In this case, the desk had rounded edges.

Materials given: 1" graph paper that has area $9" \times 12"$.

Approaches: Teacher gave them only 2 or 3 pieces of graph paper per group so that they couldn't completely cover the table (it took about 8 pieces) and had to use some reasoning. They counted that there are 108 square inches on each paper. Then they knew there were 8 papers so they multiplied 108 square inches/paper x 8 papers = 864 square inches. Some groups subtracted 2 rows of 9" that were overhanging. Some groups left their answer as the estimate 864 square inches. Most students ignored the rounded edges because it seemed to be less than one square.

FIFTH GRADE: ESTIMATION, MULTIPLICATION/DIVISION, STRATEGIC THINKING

Original question: Victor needs to pack up some books for storage. The table shows the weight of the different types of books that Victor needs to pack.

BOOK TYPE	TOTAL WEIGHT
Math	55 pounds
Reading	86 pounds
Science	74 pounds

TABLE B.2: BOOK WEIGHTS

Each box can hold 30 pounds. What is the fewest number of boxes he will need?⁷

- A. 3 boxes
- B. 7 boxes
- C. 8 boxes
- D. 9 boxes

Modeling question: Look at this bookshelf in the class. Make a plan for packing it in boxes, so you use the least number of boxes, but the boxes are not too large to carry. You don't have time to measure or weigh each book.

SIXTH GRADE: PROPORTIONAL REASONING

Original problem: A 40 pound bag of adult dog food costs about \$35.00. If a serving is $1\frac{1}{4}$ pounds per day and you have 3 dogs, how long will that bag of dog food last?

Modeling problem: How much dog food would we have to buy to feed all the stray dogs

in the world for one year? How much would it cost? How much does this compare to the cost to feed all of the hungry children in the world for one year?

SEVENTH GRADE (SBAC ASSESSMENT): CONVERSION OF UNITS, AREA, DIVISION, CALCULATOR

Original problem: You are tiling a $5' \times 8'$ rectangular kitchen floor and you are tiling with 3.5" square tiles that come with 60 tiles in each box. What is the least number of boxes you need to buy? If necessary you could cut the tiles to fit the room.⁸

Modeling problem: You are tiling a $5' \times 8'$ rectangular kitchen floor and you have a choice to use 3.5" square tiles or 4.25" square tiles. If necessary you could cut the tiles to fit the room, but it wastes a lot of time and sometimes they break. Which shape tiles would you use if they cost the same? Which would you use if the 3.5" tiles cost half as much but break more? Make a mathematical argument.

EIGHTH GRADE: EXPRESSIONS AND EQUATIONS, SCIENTIFIC NOTATION

Original problem: Ants versus humans. The average mass of an adult human is about 65 kilograms while the average mass of an ant is approximately 4×10^{-3} grams. The total human population in the world is approximately 6.84 billion and it is estimated that there are currently about 10,000 trillion ants alive. Based on these values, how does the total mass of all living ants compare to the total mass of all living humans?⁹

Modeling problem: Consider the problem above, but to calculate the mass of all humans, model the average weight of a human (not just an adult).

DIMENSIONS	SCORE OF 0	SCORE OF 1	SCORE OF 2	SCORE OF 3	SCORE OF 4
FIRST SIX PLUS ONE MINI GOLF HOLES ARE CONSTRUCTED USING GEOGEBRA	Not completed at all	Mini golf course holes are constructed with major issues	Mini golf course holes are constructed with minor issues and no practice hole is included	Mini golf course holes are constructed with minor issues	Mini golf course holes are well constructed
COORDINATES OF CORNERS OF HOLES AND CUP ARE IN THE REPORT	Coordinates not included at all	Coordinates of all holes are included, but not even close to correct	Coordinates of all holes are included with major issues	Coordinates of all holes are included with minor issues	Coordinates of all holes are perfect
PERIMETER	Perimeter not included at all	Perimeters of all holes are included, but not even close to correct	Perimeters of all holes are included with major issues	Perimeters of all holes are included with minor issues	Perimeters of all holes are perfect
AREA	Areas not included at all	Areas of all holes are included, but not even close to correct	Areas of all holes are included with major issues	Areas of all holes are included with minor issues	Areas of all holes are perfect
PARALLEL LINES AND ANGLES	Parallel lines and angle measures not included at all	Parallel lines and angle measures of all holes are included, but not even close to correct	Parallel lines and angle measures of all holes are included with major issues	Parallel lines and angle measures of all holes are included with minor issues	Parallel lines and angle measures of all holes are perfect
COMPOSITION OF POLYGONS AND SOLIDS	Composition of polygons and solids not mentioned	Composition of polygons and solids mentioned, but not even close to correct	Composition of polygons and solids described, but with major errors	Composition of polygons and solids described with only minor errors	Composition of polygons and solids described perfectly
SURFACE AREA	Surface Areas not computed	Surface Areas computed, but not even close to correct	Surface Areas computed, but with some major errors	Surface Areas computed with only minor errors	Surface Areas computed perfectly
VOLUME	Volumes not computed	Volumes computed, but not even close to correct	Volumes computed, but with some major errors	Volumes computed, but with only minor errors	Volumes computed perfectly

DIMENSIONS	SCORE OF 0	SCORE OF 1	SCORE OF 2	SCORE OF 3	SCORE OF 4
REGULAR POLYGON ANGLE MEASURES	Regular Polygon angles not computed	Regular Polygon angles computed, but not even close to correct	Regular Polygon angles computed, but with major errors	Regular Polygon angles computed with only minor errors	Regular Polygon angles computed perfectly
AREA OF WALKWAYS	Areas of walkways not computed	Areas of walkways computed, but not even close to correct	Areas of walkways computed, but with major errors	Areas of walkways computed with only minor errors	Areas of walkways computed perfectly
CARPETING	Tessellation is not discussed at all.	Tessellation is discussed, but is not even close to correct	Tessellation is discussed with major errors	Tessellation is discussed with minor errors	Tessellation is discussed perfectly
SECOND SIX PLUS ONE MINI GOLF HOLES ARE CONSTRUCTED USING GEOGEBRA	Not completed at all	Mini golf course holes are constructed and labeled with major issues	Mini golf course holes are constructed and labeled with minor issues and no practice hole is included	Mini golf course holes are constructed and labeled with minor issues	Mini golf course holes are well constructed and labeled
TRANSFORMATIONS FOR SECOND DESIGN	Transformations not described	Transformations described, but not even close to correct	Transformations described, but with some major errors	Transformations described but with only minor errors	Transformations described perfectly
COST	Costs not computed	Costs computed, but not even close to correct	Costs computed, but with some major errors	Costs computed with only minor errors	Costs computed perfectly
REPORT	There is no report	Report is not well organized and does not convey information well	Report is organized but does not convey information well	Report is not organized but conveys information well	Report is organized and conveys information well
				SCORE OUT OF 60	

TABLE B.1: SCORING RUBRIC FOR THE MINI GOLF PROJECT

C

1

APPENDIX C: EXTENDED EXAMPLES

MODELING EXEMPLARS

This section contains four extended problems each illustrating, in some detail, the modeling process and each focusing on a different component in the modeling cycle.

The first problem is the *Elevator Problem* (Problem C.1). This presents teachers with examples of variations of the same problem, with different levels of scaffolding, for students who are new to modeling and those with experience. It has been noted that the problem statements for true modeling experience should be as open as possible, allowing students many approaches. However, students early in the process struggle with the open-ended nature of modeling, so finding a way to support students without taking away the adventure is essential. Fortunately, there is no need to have different problems for beginners than for experienced modelers. The same problem scenarios are open for investigation at all levels of experience, but the support given to the students can be tuned to their mathematical tools and modeling experience.

The second example is the *Midge Problem* (Problem C.2). Students need to find a way to distinguish two different species of biting insects based on their wing and antennae lengths. Similar to the previous problem, this is presented as an open-ended investigation that lays out one example of the back-and-forth among the components of the modeling cycle as students create, assess, and improve their models. One of the distinctive features of modeling that separate it from everything that students are familiar with—the iterative process of improving and refining the model—is illustrated so that readers may experience the modeling thought process.

The third example is *Distributing Disaster Relief Funds Fairly*. Students must consider a variety of methods for distributing too little money to requests that all have merit. Different definitions of fair are posed and students must decide how to implement the method and

assess its value relative to other methods being proposed. Historical aspects of the problem are also presented, since the issues of fair division reach far back in human history.

The fourth example is *Driving for Gas*, which was first presented in a slightly different form in Chapters 1 and 3. *Driving For Gas*, presents a view of modeling as a classroom activity in which students work together in small groups with the teacher supporting their efforts. The example is designed for students in Algebra or Precalculus who are relatively new to mathematical modeling, so there is a great deal of scaffolding and support for the process by the teacher in the presentation.

THE ELEVATOR PROBLEM

The *Elevator Problem* can be used with many different classes. Naturally, the more advanced the class, the more that can be expected in the solution. The problem has no mathematical prerequisites and has been used successfully with Algebra 1, Precalculus, and post-Calculus students. After spending several class periods on the problem, one Precalculus student commented, “Never before, in all my math experiences, had I seen a problem as open ended and varying as this one. Working on a problem like this with no obvious answer and many different options was a wholly new experience for me. This problem helped me visualize the role math could and most likely will play in my future.”

If used as a motivation or application of standard content, an algebra class could focus on developing a simple linear function of two variables, while a class that has studied some probability might focus on the problem as an application of combinatorial probabilities. It can also serve as a vehicle for discussing simulation models involving probability. The *Elevator Problem* serves equally well as a totally open problem unrelated to the current content being studied. Problems like this one demonstrate that modeling at the high school level often involves a sophisticated application of very elementary concepts.

The problem focuses attention to the importance of making reasonable, simplifying assumptions. Like many modeling problems, the *Elevator Problem* can be made as simple or complex as the teacher desires by altering a few of the parameters in the problem and by the amount of scaffolding and student–teacher conversation in the presentation of the problem. Each teacher can determine how difficult the problem should be and how far into the problem they want their students to go, based on their goals for the activity.

Problem Statement Walton and Davidson in the Spode Group’s wonderful volume *Solving Real Problems with Mathematics, Volume 2* present the elevator problem in the form of a series of memos between your boss and you (the student), and between you and your assistant discussing the problem of late arrivals at work.

MEMO #1

From: Your Boss

To: You

Re: Late Arrivals

I have received numerous complaints that large numbers of our employees are reaching their offices well after 9:00 a.m. due to the inability of the present three elevators to cope with the rush at the start of the day. In the present financial situation it is impossible to consider installing any extra elevators or increasing the capacity of existing ones above the current ten persons. Please investigate and let me have some possible solutions to the problem with an indication of their various advantages and disadvantages.

MEMO #2

From: You

To: Your Assistant

Re: Late Arrivals

Can you find out:

1. How long the elevators take to get between floors and how long they stop for?
2. How many people from each floor use the elevator in the morning?
3. How many people were late this morning?

MEMO #3

From: Your Assistant

To: You

Re: Answers to your questions

1. The elevators appear to take 5 seconds between each floor, an extra 15 seconds for each stop, and another 5 seconds if the doors have to reopen. It also seems to take about 25 seconds for the elevator to fill on the ground floor.
2. The number of workers on each floor are:

FLOOR	G	1st	2nd	3rd	4th	5th
NUMBER	0	60	60	60	60	60

3. About 60 people were late today.

MEMO #4

From: You

To: Your Boss

Re: Solution to the problem with advantages and disadvantages

?

In the problem posed, we see that the building has 5 floors (1-5) that are occupied. The ground floor (0) is not used for business purposes. Each floor has 60 people working on it and there are 3 elevators (A, B, and C) available to take these employees to their offices in the morning. Each elevator holds 10 people and takes approximately 25 seconds to fill on the ground floor. The elevators then take 5 seconds to travel between floors and 15 seconds on each floor on which it stops.

Have students discuss the situation and think about what might be creating the late arrivals and how the elevator system might be changed to address the problem. As one aid, you might have them discuss the elevator signs from a hotel (Figure C.I).

Could such a system be utilized during the morning rush (8:30 am - 9:00 am)?

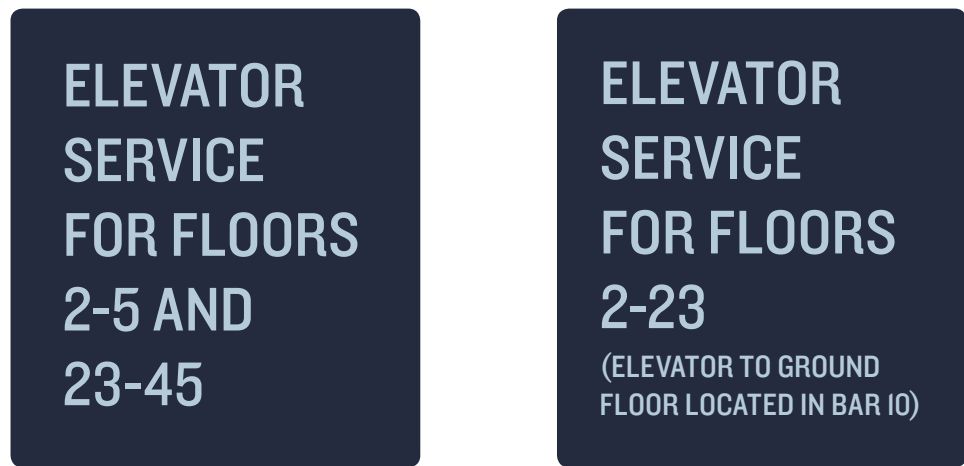


FIGURE C.I: HOTEL ELEVATOR SIGNS AS AN EXAMPLE ELEVATOR SYSTEM

MAKING REASONABLE SIMPLIFYING ASSUMPTIONS

Before proceeding to a discussion of the problem, it will be helpful to look at some basic assumptions that can be made to simplify and clarify the problem to be solved.

Students must make some assumptions about the current process to set up their work. The assumptions need to be couched in the reality of the problem setting, but allow for some simplicity in modeling the process. The assumptions must also be consistent with, or at least not contradict the information at hand.

Students will often try to solve the problem by requiring employees to arrive at specific times (those working on the fourth floor should arrive at 8:47, for example), or requiring those working on the lower floors to take the stairs, but to the extent possible, we would like to alter the elevators, not the habits of the individuals. Encourage the modelers to make no changes to the worker's current morning routine. Changing the daily patterns of

the employees will create problems between the workers and management that should be avoided. This charge means we need to make some assumptions about what that current routine could likely be.

For example, we could not assume all employees arrive at 5 minutes to 9:00, since, if that were the case, we would have many more than 60 employees late. Since the boss did not write to complain about employees arriving at the elevators too late to get to work on time, we can assume that employees arrive at a time that should allow them to get to work, but operation of the elevators keeps them from it.

Note: Have students discuss plausible reasons for workers arriving late for work because of the slowness of taking the elevators.

Assumption 1: We assume that at some time prior to 9:00, employees begin arriving, and, once the arrival process starts, there is a steady stream of employees waiting to take the elevators. The memo does not mention workers arriving at the elevators late, but arriving in their offices late. This assumption gives us a realistic problem that might have a solution involving the elevators.

Assumption 2: We assume that the current situation is that each elevator carries employees to all floors, necessitating stops on each floor on each trip. When in a hurry to get to their office, we expect that the workers will take the first available elevator. This creates a mixed group filling each elevator, which may be the reason the trips take too long.

Assumption 3: Since the issue is getting employees to their floor efficiently, we assume that the only use of the elevators between 8:30 and 9:00 is in getting from the ground floor to the appropriate floor for their job. We will ignore the possibility that workers will be moving between floors (going from 4th to 2nd, for example) during this time.

While this assumption is clearly unrealistic, we have no information about how prevalent this movement is and consequently no good way to incorporate it in the model. Moreover, trying to do so initially makes the problem much too complicated. Once students have created an initial model, they may be able to add this piece of the problem to it, but it is very important that students keep their first models as clean and simple as possible. The modeling cycle allows for successive refinement to deal with smaller, but still important components that have been initially ignored. Our first model will focus only on moving employees from the garage up to their floor.

Assumption 4: Elevator doors do not re-open. Since everyone is anxious to get to work on time, they exit the elevators efficiently. In a first model, considering small issues like this can make the problem more difficult and can hide some important features.

A SIMPLE MODEL OF ELEVATOR MOVEMENT

One way to approach this problem is to consider the worst case scenario. The worst that can happen is for each elevator to always have at least one person from each floor on it. This means the elevator will have to make the longest possible trip each time. Since there are 60 people on 5 floors, there are 300 employees to take to their offices. On average, we would expect that each of the elevators will carry 100 people. Since the capacity of each elevator is ten people, each elevator will make 10 trips. How long will each of these trips take? The Figure C.2 below illustrates one complete trip. Since no one is being picked up or let off, the elevators do not stop on the way down.

The total time of the trip is given by $T = 25 + 5(10) + 5(15) = 150$ seconds per trip.

Since each elevator makes 10 trips and there are three moving simultaneously, the total time is 1500 seconds, or about 25 minutes. Some students may note that we don't need to count the final trip back to the ground floor, and use 1475 seconds as their estimate.

In the calculation above, we have made an important assumption. If we have a crowd of workers waiting for the elevators, then when the elevator door opens on the ground floor, there are 10 employees ready to get on. Each elevator is full at the start of its ride.

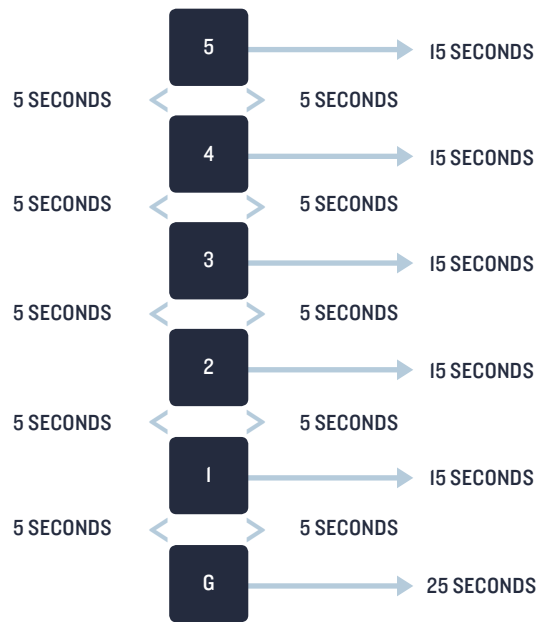


FIGURE C.2: GRAPHICAL MODEL OF AN ELEVATOR TRANSIT.

Assumption 5: Every elevator is full with 10 workers getting on at the ground floor.

Note: Once we have a solution to the nice problem using 10 people per trip, we can recalculate everything using an average of 8 or 9 to see how our model performs. But we want to keep the first model neat and clean to see what the important issues are.

MODELING THE CURRENT STATE

We know that 60 workers were late this morning. Sixty workers represents six elevator trips under our assumptions. With three elevators moving simultaneously, each elevator must make two trips after 9:00. The total time to move all 60 workers to their floor under our initial assumptions is $(2 \text{ trips})(150 \text{ sec/trip}) = 300$ seconds, or 5 minutes. We need to cut around 300 seconds from the 1500 seconds our simple models predicts it takes to move everyone upstairs. Our target is a maximum of 1200 seconds. Is this possible?

Since it takes approximately 1500 seconds for everyone to be taken to their offices, and your assistant found that 240 (80%) of the employees arrived on time, we can conclude that people begin arriving at a time that allows only 80% of the required 1500 seconds to occur before nine o'clock. That is, they begin to arrive approximately 20 minutes before 9:00.

Rerouting the Elevators It was suggested earlier, in the discussion about the hotel elevators, that rerouting some of the elevators during the rush might be helpful to the company. Let's try one example: what would happen if two elevators went to floors 1, 2, and 3, and one elevator went to floors 4 and 5? How long would this configuration take to get everyone to the proper floor?

We could use the diagram above to model the situation for each elevator, or we can create an algebraic expression whose value gives the time of the trip more simply.

Have the students discuss what the total transit time of an elevator depends upon. What determines how long it takes? Students will recognize that the transit time depends upon two things, the number (N) of floors at which the elevator stops, and the highest floor (F) to which the elevator travels. The transit time is

$$T = 25 + 10F + 15N \text{ seconds per trip.}$$

In the example given above, the trip to the first, second and third floors takes

$$T = 25 + 10(3) + 15(3) = 100 \text{ seconds per trip,}$$

while the trip to the fourth and fifth floors takes

$$T = 25 + 10(5) + 15(2) = 105 \text{ seconds per trip.}$$

As shown in Table C.1, with this configuration, the slowest elevator takes 1260 seconds, a

reduction in total transit time of 4 minutes over having all elevators travel to all floors. Can we do better? Can we reduce the expected time by our goal of six minutes?

As shown below, if we let one elevator travel to floors 1 and 2 and the other two travel to floors 3, 4 and 5, then we can reduce the total transit time to 1080 seconds, or 7 minutes less than having the elevators travel to all floors. That meets our goal, but are there even better arrangements?

The problem can be stopped here if desired. The focus of the problem may be on setting the simplifying assumptions, generating the time equation $T = 25 + 10F + 15N$, and using it in an interesting, problem solving investigation. This problem lends itself well to group work, as students working in groups can quickly investigate other promising combinations of elevators. We based our calculations on the worst case assumption that every elevator stops at every floor to which it can travel and the best case assumption that every elevator would be full. We have a solution that does not require the employees to alter their preferred arrival time, only to take a specific elevator between 8:30 and 9:00 in the morning. After 9:00, the elevators are available for use by everyone.

ELEVATORS	FLOORS SELECTED	PEOPLE CARRIED	TRIPS REQUIRED	TIME/TRIP (SECS)	TOTAL TRAVEL TIME (SECS)
A & B	1, 2, 3	180	9 each	100	900
C	4, 5	120	12	105	1260

TABLE C.1: TOTAL TRANSIT TIME WITH ELEVATOR C TO FLOORS 4 AND 5.

ELEVATORS	FLOORS SELECTED	PEOPLE CARRIED	TRIPS REQUIRED	TIME/TRIP (SECS)	TOTAL TRAVEL TIME (SECS)
A	1, 2	120	12	75	900
B & C	3, 4, 5	180	9 each	120	1080

TABLE C.2: TOTAL TRANSIT TIME WITH ELEVATOR A TO FLOORS 1 AND 2.

Now that students see how the problem works, they can include such details as having an average of 9 workers per trip, consideration of trips that, just by chance, don't stop on every floor, or other more realistic components. We will consider some variations of our assumptions later.

PRESENTING THE PROBLEM

The problem as described so far, has been designed for students new to modeling. This problem serves well as a first modeling adventure, so a whole-class conversation setting the stage and supporting the development of assumptions is helpful in getting the students on a reasonable track.

As mentioned in *Driving for Gas*, asking specific questions can help the student focus on important issues. The goal of this questioning is not to tell students what to do, but to raise questions that, as students think about and discuss them, illuminate the important issues for the students. It is a form of scaffolding for students. For example, the list of questions below has been used in an all-class discussion with Precalculus students who were engaging this simple version of the *Elevator Problem* as their first mathematical modeling experience. The goal of the questioning is to have students think about and come up with the important considerations to be addressed in the model. It is important that the questions not be overly directive.

- In some buildings, all of the elevators can travel to all of the floors, while in others, the elevators are restricted to stopping only on certain floors. Why?
- What is the advantage of having elevators which travel only between certain floors?
- Suppose it is a holiday and only 5 people come to work today. Each person works on a different floor, and they all ride the same elevator. How long will it take for everyone to get to work?
- Now suppose that 5 people come to work and these five people do not all work on different floors. How long will it take for everyone to get to work?
- Why was the last question harder to answer than the prior question? What assumptions will you need to make in order to simplify the problem?

Such questions can convince students to make some reasonable simplifying assumptions about how the elevators fill on the ground floor. Typical student assumptions include:

- There will be 10 people waiting for the elevator on the ground floor and the elevator will fill to capacity (10 people) for each trip.
- If an elevator can go to a floor, then it will go to that floor on each trip.
- No one takes the stairs.
- No one uses the elevator to go down during this time (or if they do, it does not impact the time for the elevator to complete its trip).
- The elevator doors don't have to re-open on any floors.

These are the assumptions used in the basic solution presented above. The first two assumptions will seem unrealistic to students, but the goal of assumption is to simplify the problem to a tractable form given the abilities of the students. Both of these assumptions can be modified once the initial simple problem has been solved. This is a difficult issue for some students. As one student said in his evaluation of the problem, “It is difficult for us to let go of the details”. Students may initially feel like making assumptions is cheating somehow. Once through a modeling cycle and they see that they can return to the assumptions and modify them to make them more realistic, they accept that good assumptions can be unrealistic in their first instance.

After agreement is reached on the basic assumptions to be used, some new thought-provoking questions might be asked.

- If all elevators go to all floors, how long will it take everyone to get to work?
- If 80 people were late using the unrestricted elevators, approximately what time did the employees begin arriving at the ground floor?

Once this last question is answered, the students have a target for their task. Beginning at the same arrival times, students need to reduce the total time to get everyone to the appropriate floor by however long it takes for the 80 late employees to reach their destination.

- Reassign the elevators to transport the employees to their offices as quickly as possible. What arrangement produces the shortest time? If this arrangement had been used today, would everyone have arrived at their floor on time?

Once completed, the students can be led to reflect on their work and comment on the strategies they use that could be applied to future mathematical modeling adventures.

Examples include:

- We used a simple case to understand the structure of the problem.
- We drew a diagram to help us visualize the scenario.
- We thought about what made the problem hard to help us figure out simplifying assumptions.
- We considered the *worst case scenario* (if an elevator can go to a floor, it will go to that floor) and solved this rather than trying to think about all of the different possibilities.
- We found a solution that worked, then modified it to see if we could improve it.
- We had to make sure our solution was realistic. (Sometimes “mathematically optimal” is not optimal in the real world.)

This last comment concerning mathematically optimal solutions is important. Students can easily see that having all workers on a floor arrive at a specific time and all ride the same elevator, so it only makes one stop, would be much more efficient. They can create a much “better” result than the one described above. But, that solution would not work well in the real world.

As mentioned earlier, each teacher can determine how difficult the problem should be and how far into the problem they want their students to go, based on their goals for the activity. One attribute of good modeling problems is that they can be extended in many directions if students have the necessary mathematics in their background. Variations of this problem have been used successfully with students from Algebra I to post-Calculus students in AP Statistics.

SECOND ITERATION AND IMPROVING THE MODEL

The full elevator assumption may be true at the end of the process when everyone is eager to get to work, but it may not be true early in the process. Suppose, on average, there are only 9 workers riding each elevator. In this case we have a couple of extra trips that are required. This is an easy change to make, now that we see how the process works.

The total time of the trip is still given by
 $T = 25 + 5(10) + 5(15) = 150$ seconds per trip.

Now there will be two elevators making 11 trips and one making 12 with only a few workers on the last trip (see Figures C.3 and C.4). Students may decide to ignore the final trip. If we can reduce the number of late workers from 60 to 4 or 5, we have essentially solved the problem. Management will likely not be too concerned about several workers arriving a minute after 9:00. Using the formula found earlier, students find the 11 trips take 1650 seconds (27.5 minutes) and the last elevator takes 30 minutes. Since 60 workers were late, this is three trips or 450 seconds. Our new target is around 1200 seconds.

The solution with one elevator going to floors 1 and 2 and the others to 3, 4, and 5 continues to work in this setting. If we reduce the average number to 8 per trip, then we have 12 or 13 trips per elevator (see Figure C.5). The time for these trips is between 30 and 33 minutes. Our new target is around 24 minutes or 1440 seconds.

ELEVATORS	FLOORS SELECTED	PEOPLE CARRIED	TRIPS REQUIRED	TIME/TRIP (SECS)	TOTAL TRAVEL TIME (SECS)
A & B	1, 2, 3	180	10 each	100	1000
C	4, 5	120	14	105	1470

TABLE C.3: TOTAL TRANSIT TIME WITH ELEVATOR C TO FLOORS 4 AND 5.

ELEVATORS	FLOORS SELECTED	PEOPLE CARRIED	TRIPS REQUIRED	TIME/TRIP (SECS)	TOTAL TRAVEL TIME (SECS)
A	1, 2	120	12	75	900
B & C	3, 4, 5	180	9 each	120	1080

TABLE C.4: TOTAL TRANSIT TIME WITH ELEVATOR A TO FLOORS 1 AND 2.

ELEVATORS	FLOORS SELECTED	PEOPLE CARRIED	TRIPS REQUIRED	TIME/TRIP (SECS)	TOTAL TRAVEL TIME (SECS)
A	1, 2	120	15	75	1125
B & C	3, 4, 5	180	12 each	120	1440

TABLE C.5: TOTAL TRANSIT TIME WITH ELEVATOR A TO FLOORS 1 AND 2.

We just barely make it. If the average was reduced to 7 per trip, which seems unrealistically low, then our solution would not work (but perhaps another variation would). By recomputing with nine, then eight, passengers per trip, we are testing the sensitivity of our model to the assumption that each elevator is full. In this case, as long as there are at least 8 passengers, on average, per trip, our solution will work. We say that our model is fairly insensitive to a change in this assumption.

THIRD ITERATION AND PROBABILITY

How realistic is our worst case assumption that an elevator will go to every floor possible? For example, how likely is it that an elevator with 10 passengers carries none of the 60 employees working on the fifth floor?

Since all of the floors have the same number of workers and thus, are equally likely on an elevator, students can easily simulate this probability with their calculators. If each student uses the random integer command to create 10 random integers from 1 to 5 (`randInt(1,5,10)` on the TI-84), they can simulate the floors that 10 workers in an elevator will go to. If they look through the ten integers to count how often none of them are a 5, they should see, by repeating the process 20 times and comparing results, that about 1 in 10 trips has no one going all the way to the 5th floor. So, instead of taking 150 seconds, this trip would take only 125 seconds. Since there are equal numbers of employees on each floor, the results for those floors are similar.

If students have studied probability, they could do a binomial approximation consistent with the random integer simulation by computing the probability that all 10 workers in an elevator come from floors 1-4 by $(4/5)^{10} \approx 0.107$.

If they have studied combinations, they could compute the actual probability as

$$\frac{\binom{240}{10} \binom{60}{0}}{\binom{300}{10}} \approx 0.103$$

Using simulations or calculations, students can expect that of the ten trips an elevator makes, one of them would go only to the first 4 floors and take only 125 seconds for the trip. In a similar manner, we find that 10% of the transits don't stop on floor 1, 10% don't stop on floor 2, 10% don't stop on floor 3, and 10% don't stop on floor 4. Each of these transits takes 135 seconds. This analysis is also not quite right, since it only applies to the first elevator trip. Once the first trip is made, the number of workers on each floor changes. But this gives a reasonable approximation that can be useful.

More advanced students might also consider if some of the 10% that don't stop on the 5th floor also don't stop on the 4th. Simulations will show that in 10 trips, this happens so rarely that it need not be considered. The probability that an elevator with 10 people has no one from either the 4th or 5th floors is

$$\frac{\binom{240}{10} \binom{60}{0}}{\binom{300}{10}} \approx 0.005$$

This means that of the 10 trips, we would expect none of them to miss two floors.

Indeed, all two floor combinations have this same probability, since all floors have the same number of people. Three floor combinations are even less likely. In 10 trips, it is reasonable to think that some of the elevator transits will miss one floor, but none will miss more than one. Of the 10 trips, we expect that five go to all floors, while the remaining five each miss one floor. So a more realistic expected total travel time is $5(150) + 1(125) + 4(135) = 1415$ seconds,

or only about 23.5 minutes. Each of the 30 elevator trips has an expected travel time of 142 seconds, so the three late trips still require a reduction of about 7 minutes. This is not much different from the first analysis. Having an unequal number of employees on the floors greatly increases the importance of these probabilistic considerations and can be used for more experienced modelers or more advanced students.

Using the two sample scenarios from our first model, we can incorporate the probability of skipping a designated floor. If an elevator is going to only three floors, carrying 90 riders, then simulations show that the probability that it skips a floor is only about 1 out of every 100 trips. Since each elevator only makes 9 trips, it is unlikely to be seen. For an elevator visiting only two floors, the probability of skipping a floor is insignificant. The result is that the expected travel times are the same, but our target has gotten a bit smaller. As long as there are equal numbers of workers on each floor, we do not lose much in our model by considering the worst case travel time. In many office buildings, the upper floors have larger offices and meeting rooms, so fewer employees work there. In these buildings, it can be quite common for elevators to not travel to all floors.

For many classes, these probability considerations are well beyond the abilities of the students and the interest of the teacher. However, it is a nice application of probability either through simulation or computation for students in classes which consider probability models.

FOURTH ITERATION IF DESIRED

As noted earlier, the problem could easily stop after the first model. More experienced modelers should consider testing the always full assumption and do some simulations for a more realistic travel time. If students have done both, they should consider combining the average of 9 per trip with the expected number of floors skipped to produce a final model that captures more of the reality of the situation than did the first model. The choice of how far to take any modeling project rests with the teacher and what the goals of the project were. The problem serves well as a modeling experience, but could be used to motivate the probability simulations or as an opportunity to use them if they had been previously learned.

ADDITIONAL PROBLEM VARIATIONS FOR MORE ADVANCED/EXPERIENCED STUDENTS

Teachers can increase or decrease the difficulty of the problem by changing the number of elevators and floors. Having unequal populations on the floors significantly increases the challenge offered by the problem. Adding in a time constraint for doors having to reopen when the elevator is crowded changes the time equation. Students need to determine in what situations will the door likely reopen.

For an advanced class, I suggest the following employee distribution:

MEMO #3

From: Your Assistant

To: You

Re: Answers to your questions

1. The elevators appear to take 4 seconds between each floor, an extra 10 seconds for each stop, and another 5 seconds if the doors have to reopen. It also seems to take about 15 seconds for the elevator to fill on the ground floor.

2. The number of workers on each floor are:

FLOOR	G	1st	2nd	3rd	4th	5th	6th
NUMBER	0	80	80	40	80	20	20

3. About 70 people were late today.

In this scenario, there will be 32 elevator trips if all elevators are full. Clearly, with only 20 employees on the 5th and 6th floors, not all of them will stop at each floor. In this setting, the probability computations are essential. Also, with different number of workers on the floors and with the addition of a 6th floor, there are more combinations of elevator trips for students to consider. Another variation is to add a fourth elevator to again increase the number of options students should consider.

**THE MIDGE
PROBLEM**

In this problem we illustrate several cycles of a modeling cycle. While it is important for students beginning to model in mathematics to have shorter, smaller components of a modeling cycle to develop their skills, having experiences with the iterative aspects of modeling is essential.

In this problem students need to find a way to distinguish two different species of biting insects based on their wing and antennae lengths. In this example, the steps in a modeling cycle are highlighted to illustrate the movement back-and-forth among the components of the modeling cycle as students create, assess, and improve their models. One of the distinctive features of modeling that separate it from most mathematical activities that students are familiar with is the iterative process of improving and refining the model. For the most part, in their past experience in mathematics, once a solution is obtained, it is either correct or incorrect. Either way, the problem is done. In mathematical modeling, each solution generates a new starting position to begin again and improve upon the previous solution.

In 1981, two new varieties of a tiny biting insect called a midge were discovered by biologists W. L. Grogan and W. W. Wirth in the jungles of Brazil. They dubbed one kind of midge an *Apf* midge and the other an *Af* midge. The biologists found out that the *Apf* midge is a carrier of a debilitating disease that causes swelling of the brain when a human is bitten by an infected midge. Although the disease is rarely fatal, the disability caused by the swelling may be permanent. The other form of the midge, the *Af*, is quite harmless. In an effort to distinguish the two varieties, the biologist took measurements on the midges they caught. The two measurements most easily obtainable were of wing length and antennae length, both measured in centimeters.

AF MIDGES

WING LENGTH (CM)	1.72	1.64	1.74	1.70	1.82	1.82	1.90	1.82	2.08
ANTENNA LENGTH (CM)	1.24	1.38	1.36	1.40	1.38	1.48	1.38	1.54	1.56

APF MIDGES

WING LENGTH (CM)	1.78	1.86	1.96	2.00	2.00	1.96
ANTENNA LENGTH (CM)	1.14	1.20	1.30	1.26	1.28	1.18

Is it possible to distinguish an *Af* midge from an *Apf* midge on the basis of wing and antenna length? If so, use your method to classify three new midges with Wing and Antenna lengths of (1.80, 1.24), (1.84, 1.28), and (2.04, 1.40).

PROBLEM C.2: THE MIDGE PROBLEM

IDENTIFY AND SPECIFY THE PROBLEM TO BE SOLVED

Our goal is to find a simple but effective way to distinguish a dangerous *Apf* midge from a harmless *Af* midge, using measurements of antenna length and wing length as a basis for the decision. Once we have decided on the method, we apply it to classify the three new midges.

A natural place to begin is by thinking about the assumptions under which we build our model. The fundamental question is, what must be true in order for any model we build to be useful? For example, the 15 midges shown in the table must be in some way representative of

their species. If not, no amount of mathematics can draw useful information. Brainstorming these issues should result in a first round of assumptions for the model. More assumptions may be added later as you gain insight and, hopefully, some may be removed as you improve on the model. For the Midge Problem, what other attributes must be present for the data to be useful?

MAKE APPROPRIATE ASSUMPTIONS AND DEFINE THE ESSENTIAL VARIABLES

Assumptions

- Assume the captured midges are representative of their classification and that they can be considered a random sample of midges. We need this assumption for our model to be useful in the classification process.
- All midges are either *Af* or *Apf*. There is no third type of midge to be considered. This assumption assures us that the three unknown midges will fit into one of our two types.
- Assume that neither midge is trap-happy or trap-adverse, so the proportions captured are representative of the relative proportions in the wild. This allows us to think critically about the larger number of captured *Af* midges and decide whether we should favor *Af* midges due to its observed prevalence.
- The classifications on the captured midges are correct, that is, none have been misclassified. Obviously, if one or more captured midges has been misclassified, our model will continue that misclassification.

By looking at the data and graphs of the data, what, if anything, arouses your attention or curiosity? What aspects might be useful in developing a model? Begin by looking at some plots of the data.

From scatterplots of the two bivariate data sets, we note that the *Af* midge has, in general, smaller wings and larger antennae than does the *Apf* midge. Also, there is greater variability in the measured lengths of the *Af* midges, which might be useful. Finally, all of the observed data ends in an even digit. That's curious, but it isn't obvious what that means. Can we use some or all of these observations to capture some important distinctions between the species? If so, how can we express these observations in useful mathematical forms?

We note in the scatterplot that for both species of midge, the lengths of the wings and antennae are reasonably linearly related. We must assume that this relationship extends beyond the observed data to the populations. If so, we can use linear models to help in the classification.

- Assume the observed linear relationship between wing length and antenna length for both species can be extended to others of their species. This assumption allow us to make predictions beyond the observed data.

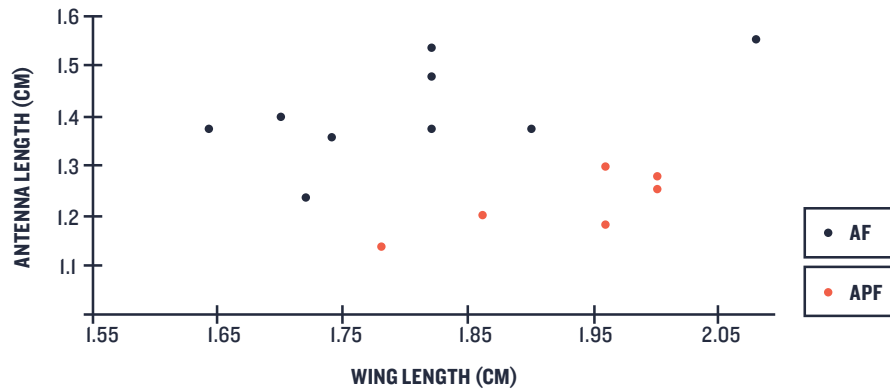


FIGURE C.3 : SCATTERPLOT OF THE TWO DATA SETS

DO THE MATH: GET A SOLUTION

From the plot in Figure C.3, we can see that there is a region between the two data sets. It seems reasonable to think that we can define a boundary which separates the two species. But where should it be located and what rationale do we have for where we place it?

We can fit a least squares line to each data set, using Wing Length as the independent variable and Antenna Length as the dependent variable. The least squares linear fit for *Af* midges is $y = 0.479x + 0.549$ while for the *Apf* midges is $y = 0.558x + 0.151$.

The best way to determine if a linear model is appropriate for a given set of data is to look at the residual plot. The residual plot is a scatterplot of the differences between the actual observed Antenna lengths and those predicted by the linear equation. So, for the *Af* midges, we plot the data indicated in the residual plots (Figure C.5).

If the plot of the residuals shows a random scatter, then the model is deemed appropriate for the data. (If the linear model captures the essence of the “trend” or “pattern” of the data, then the residuals—what’s left over when that trend is taken out—should be random “noise.”) The random scatter of the residual plots shown in Figure C.6 indicates that the linear model is plausible for each species.

ANALYZE AND ASSESS THE MODEL AND THE SOLUTION

We have created linear models for each species of midge. How can we use these two linear models to classify the two midge species. “Averaging” the two linear equations to create the mid-line between them (which splits the region in half) is a common approach for students. It seems like a natural thing to do with the two lines, but students have often never averaged equations.

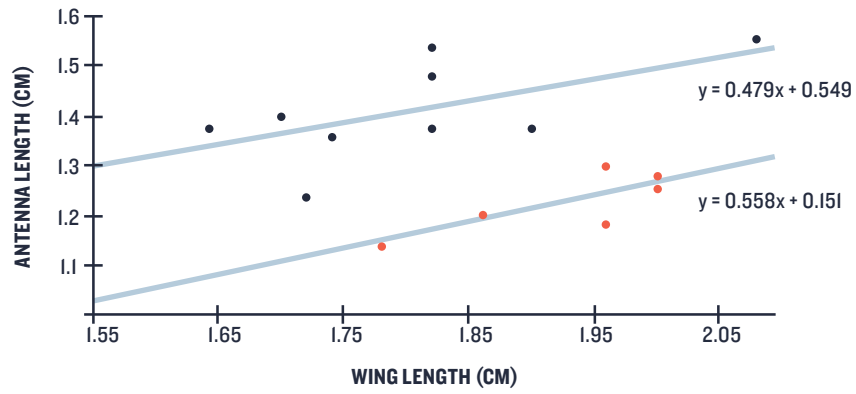


FIGURE C.4: LINEAR LEAST-SQUARES LINE FIT TO EACH DATA SET

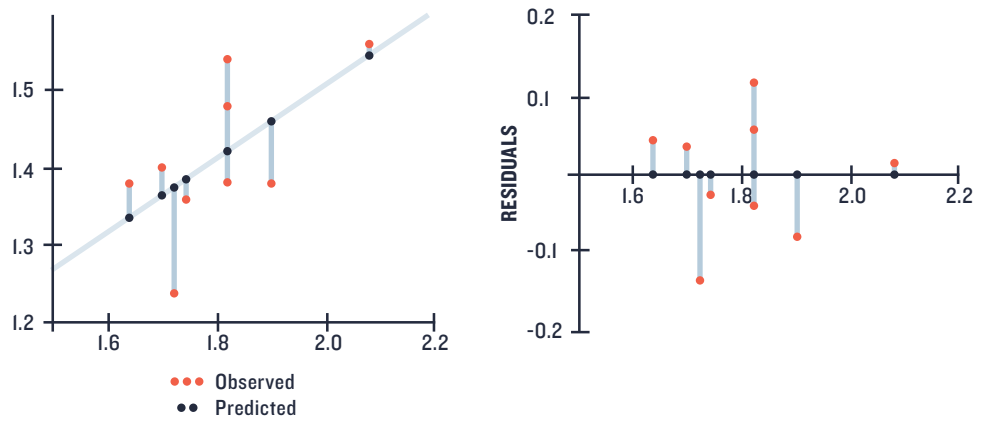


FIGURE C.5: RESIDUALS PLOTS

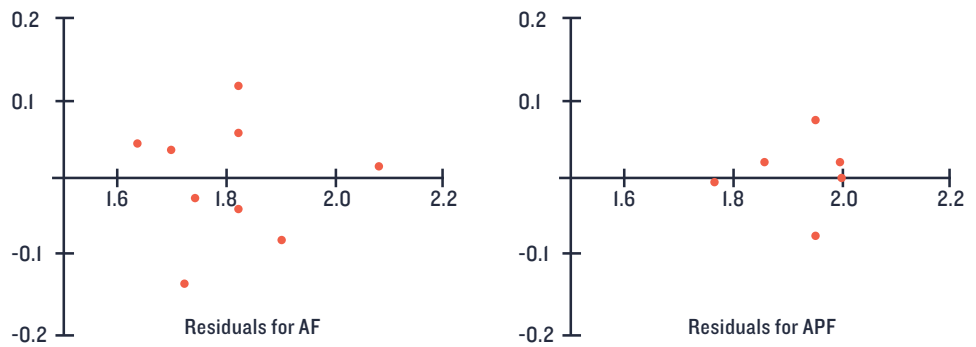


FIGURE C.6: RESIDUALS FOR LEAST SQUARES MODELS

DO THE MATH: GET A SOLUTION

To find the mid-line between these two lines, simply add the two equations and divide by two. The boundary determined in this fashion is $y = 0.5185x + 0.350$. Consider any midge below this line would be considered an *Apf* midge while any midge above the line was to be considered an *Af* midge.

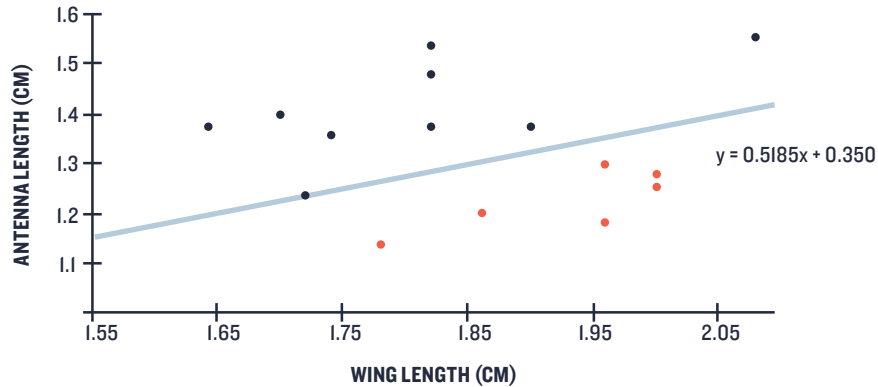


FIGURE C.7: LEAST SQUARES MID-LINE.

The line created by the mid-line of the two regression models is deficient, since it misclassifies an *Af* midge as an *Apf* midge. We have assumed all known midges were correctly classified, so we need to address this problem.

Had we intended to find a safe boundary between the species, we could argue that this misclassification was a reasonable price to pay for added safety. All of the dangerous midges have been detected, which could be the most important component. However, our problem was to find a way to distinguish the two species, not to find a boundary that would keep the biologists safe. So, we need to revise our approach.

Students might make a correction to the mid-line model, $y = 0.5185x + 0.350$. They might argue that since there are more *Af* midges than *Apf* midges in the collection, more of the region between the fitted lines $y = 0.479x + 0.549$ (*Af*) and $y = 0.558x + 0.151$ (*Apf*) should be allocated to the larger population. Since *Apf* represents three-fifths of the captured insects, it should receive three-fifths of the area. A better approach would be to weight the linear combination according to the variability of the two samples. This sample solution takes a different approach.

ITERATE AS NEEDED TO REFINE AND EXTEND THE MODEL

Looking at the scatterplot, we noticed that the *Apf* midge tends to have slightly longer wings and slightly shorter antennae than the *Af* midge. Such an occurrence might suggest that the ratio of Antenna Length to Wing Length could prove useful.

The ratios of Antenna Length to Wing Length are:

AF	0.721	0.841	0.782	0.824	0.758	0.813	0.726	0.846	0.75
APF	0.640	0.645	0.663	0.630	0.640	0.602	-	-	-

TABLE C.6: ANTENNA TO WING LENGTH RATIOS.

Figure C.8 shows these ratios on a number line.



FIGURE C.8: NUMBER LINE WITH ANTENNA TO WING LENGTH RATIOS.

There is no overlap in these ratios. The smallest ratio for an *Af* midge is 0.721 and the largest for an *Apf* midge is 0.663. Somewhere in the interval $[0.663, 0.721]$ is a boundary that might be used to effectively distinguish the two midge species. Where should we choose? If we use the midpoint, which is 0.692 (as shown in Figure C.9), we have a classification criterion that avoids the earlier misclassification error. If the ratio of Antenna Length to Wing Length is greater than 0.692, consider the midge an *Af* midge. Otherwise, consider it an *Apf* midge. Students may choose to find the means or medians of the two sets of ratios, and find the midpoint of the means or medians instead of using the extremes as shown here.

ANALYZE AND ASSESS THE MODEL AND THE SOLUTION

If $Antenna/Wing = 0.692$, then $Antenna = 0.692 \cdot Wing$ and we can view our boundary as a line in the plane.

The boundary created by the Midpoint Ratio model classifies all known midges correctly, so it is certainly an improvement. Many students will be satisfied with this solution and use it to classify the three unknown midges. However, in the plot above, it appears that the boundary is closer to the *Af* midges than to the *Apf* midges. Given the greater variation among the *Af* midges, this suggests that the Midpoint Ratio model does not effectively account for the different variability of the two types that we noted earlier. We now have a working model; perhaps we can improve upon it.

What improvements students try depends upon what mathematics they know. If they have studied statistics, they have options that would not be available to students who have not studied statistics. It is quite possible that the solution above is the best model the students can create. The variation below assumes some background in statistical models.

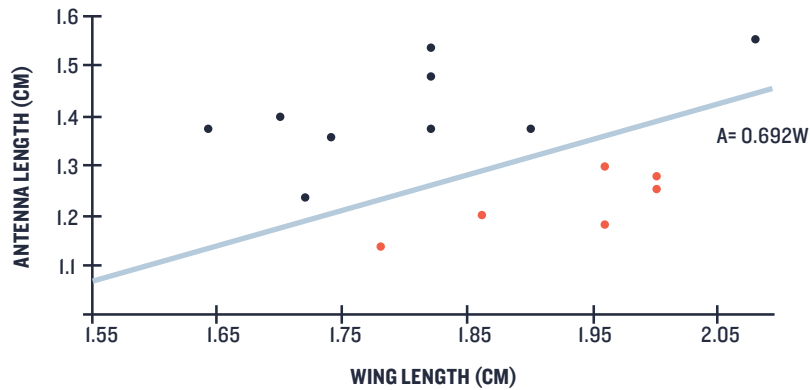


FIGURE C.9: THE MIDPOINT RATIO.

ITERATE AS NEEDED TO REFINE AND EXTEND THE MODEL

A modified ratio model: the z-score approach With technology, students can easily compute the mean and standard deviation for the two sets of ratios. For *Af* midges, the mean ratio is 0.785 and the standard deviation is 0.048, while for *Apf* ratios, the mean is 0.637 and the standard deviation is 0.020. We note that the *Af* midge ratios are more scattered than the *Apf* ratios. We can standardize the ratios by transforming them into their corresponding z-scores. A z-score, computed as

$$z = \frac{\text{observation} - \text{mean}}{\text{standard deviation}}$$

is a measure of distance from the mean, measured in units of standard deviations. The larger (or smaller for negative z-scores) the z-score, the more distant from the center is the observation. Since the z-score is based upon the standard deviation, we can include the observed difference in variability between the two species of midges by comparing their z-scores.

Note: since we don't know anything about the normality of the distributions, we can't make any statements about probabilities, but we can make comparisons. An *Af* midge with a ratio that is 0.05 units less than average has a z-score of

$$z = \frac{-0.05}{0.048} = -1.04$$

and is a little more than one standard deviation from the mean. An *Apf* midge with a ratio that is 0.04 above the average for *Apf* midges has a z-score of

$$z = \frac{0.04}{0.020} = 2$$

So, even though the *Apf* midge's ratio is closer to its average, it is a more unusual midge for its type, being two standard deviations from the mean.

Students coming out of a Common Core middle school mathematics experience might use the mean absolute deviation instead of the standard deviation. The mean absolute deviation for *Af* is 0.0368 and for *Apf* is 0.0137.

MAKE APPROPRIATE ASSUMPTIONS AND DEFINE THE ESSENTIAL VARIABLES

So, we can define a measure of unusualness with the z-scores. If two midges have equal z-scores, then they are equally unusual for their respective populations.

DO THE MATH: GET A SOLUTION

Considering z-scores allows us to find the ratio, r^* , for which the z-scores are equal. Two midges with the same z-score are considered to be equally unusual for their respective distributions. Solving

$$\frac{r^* - 0.637}{0.020} = \frac{0.785 - r^*}{0.048}$$

we find that $r^* = 0.6805$. Our ratio model suggests that an Antenna/Wing ratio greater than 0.6805 should be classified as an *Af* midge while an Antenna/Wing ratio less than 0.6805 should be classified as an *Apf* midge.

ANALYZE AND ASSESS THE MODEL AND THE SOLUTION

If $Antenna/Wing = 0.692$ is used as a boundary on the number line, then $Antenna = 0.692 \cdot Wing$ is an equivalent boundary in the plane. Should an observation fall on the boundary, for safety, we default to *Apf* and consider it dangerous. This linear boundary seems a better choice than that for the midpoint ratio (Figure C.10 and C.11), since this new boundary acknowledges and accounts for the greater variation within the *Af* population.

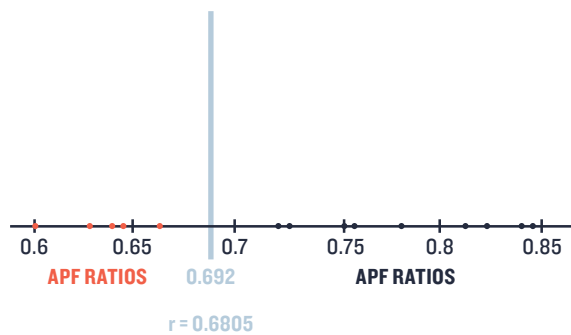


FIGURE C.10: ONE-DIMENSIONAL BOUNDARY $r^*=0.6805$

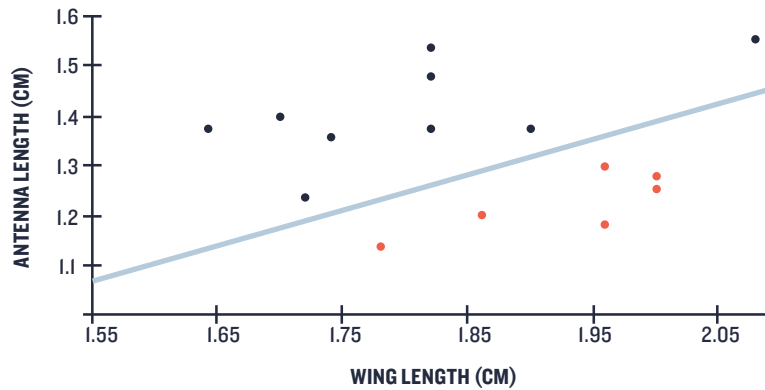


FIGURE C.11: TWO-DIMENSIONAL GRAPH $A=0.6805W$

ANALYZE AND ASSESS THE MODEL AND THE SOLUTION

A natural method for testing the model is to remove some midges, recalculate the boundary according to the process described, and see if the process would correctly classify the known midges that were removed. If a method doesn't correctly classify the midge that was removed, then it has some weaknesses. If each midge is removed one at a time and the boundary recomputed using our equi-unusual process on the remaining midges, in every case, the method predicted the type of the removed midge correctly. The slope of the line defining the boundary changed only modestly in each case.

IMPLEMENT THE MODEL AND REPORT RESULTS

Using the Equi-Unusual Ratio model, we will classify any midge with an antenna to wing ratio larger than 0.6805 as an Af midge, and those below that ratio should be considered Apf.

With this boundary we can classify the three unknown midges:

- The unknown midge at (1.80, 1.24) has a ratio of 0.689 and is classified as Af.
- The unknown midge at (1.84, 1.28) has a ratio of 0.696 and is classified as Af.
- The unknown midge at (2.04, 1.40) has a ratio of 0.686 and is classified as Af.

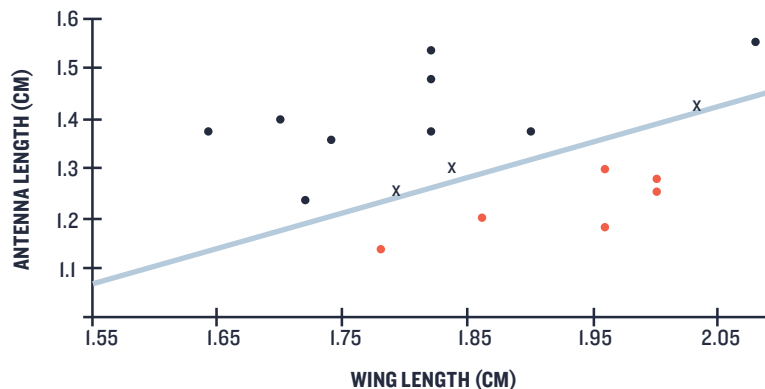


FIGURE C.12: CLASSIFYING THE THREE UNKNOWN MIDGES

In each iteration of the modeling process, a model was developed and assessed. Where it was found wanting, additional assumptions were used to refine and improve the model. Some models were abandoned, while others extended. That final model was assessed and though, like all models, it had some limitations, it was found to be satisfactory and could serve until more data can be obtained and the model improved upon once again.

INTRODUCTION

Unfortunately, natural and industrial disasters occur with distressing regularity. There are hurricanes, earthquakes, floods, tornados, industrial accidents, plane crashes, etc. These events cause individual people the loss of property, personal suffering and sometimes loss of lives.

In the wake of a storm such as a hurricane, local or national governments (or private organizations) may set aside a dedicated “pot” of money (or clothing, tents, food, etc.) to help the people or communities affected by the storm with rebuilding or dealing with losses. Typically, the fund consists of money put aside to ameliorate the disaster. However, usually the size of the fund is not large enough to cover all of the legitimate claims that are made against the sum of money allotted to try to relieve suffering.

GENERAL MODELING SITUATION

A natural disaster (e.g. hurricane) has caused extensive damage in a localized area. A special monetary fund of size E has been created for the benefit of those who suffered losses due to the disaster. Suppose that honest claims of size C_1, C_2, \dots, C_r from claimants 1, 2, ..., r have been received by the “administrator” charged with distributing the money from the special fund. Numbers are being used to indicate the names of the claimants, and C_i is the amount claimed by claimant i . Unfortunately, the total amount of the claims is larger than the amount E available to pay off the claims, so $C_1 + C_2 + \dots + C_r > E$.

Task: Devise a fair system for the fund administrator to distribute the amount E to the claimants.

Note: A variety of specific instances of this kind of problem are given in the section below. These specific instances will not only provide a test for your attempt to treat the claimants fairly but also for how to carry out your model for specific values of the claims.

Issues for you, the modeler, to keep in mind:

- The claimants in different settings may be a single person, a household, or a whole city.
- Claimants may be rich or poor, or if the claimants are cities, have very different sizes in terms of wealth, economic prosperity, physical area and/or population.
- The size of the claims in the test instances for your model are merely listed as numbers.

Thus, you can't tell from the number 6 whether this is 600,000 dollars or 6 million dollars, or 60 dollars. Does your view of what constitutes fairness change with an altered scale for the amount of money involved in the claims?

- When relief from the fund is given to a claimant i , in evaluating how to value what the claimant has received back the claimant may think in terms of what was received (gain) or may view what was "lost." Does how to view gain versus loss in the way claims are settled change your way of thinking about what is fair?
- Can you think of other situations where there is an "estate" E and the claims against E exceed the value of E ? How do these other situations resemble distributing disaster recovery funds, and how are they different?

INSTANCES FOR STUDENTS

The only information you have available is:

- Estate $E = 210$ with two claims:

- > A claims 60
- > B claims 210

Suppose, instead, $E = 40$ and the claims remain the same.

- Estate $E = 100$ with four claims:

- > A claims 40
- > B claims 40
- > C claims 60
- > D claims 160

Suppose, instead, $E = 150$; $E = 210$; and $E = 280$, and the claims remain the same in each instance.

- Estate $E = 300$ with four claims:

- > A claims 10
- > B claims 20
- > C claims 30
- > D claims 440

Suppose, instead, $E = 40$; and $E = 400$, with the claims remaining the same.

FOR TEACHERS

Here are some ways to guide students to some of the ideas that have been used to solve problems of this kind. This particular class of problems is known as bankruptcy problems. One version of how to "set" such questions is that a firm has gone bankrupt with remaining assets E but the claims against these assets from creditors exceeds E . The reason why E is often used in bankruptcy problems is that in other settings of this problem we have an estate E and the claims against this estate are larger than the amounts of money available in the estate.

Entity equity One approach might be to say each claimant should be treated as an entity, and that one should just divide the amount E equally among all of the claimants. This is the approach to giving each state in the US Senate two seats for each state no matter how large or small in population or area. Each state is treated equally.

However, this approach in some particular instances of a bankruptcy problem does allow the possibility that a claimant is given more than it claims. This will seem all right to some students but unreasonable to others. So one can modify entity equity to equalize what is given to each claimant as much as possible, but not give any claimant more than that claimant asks for. This “constrained equality of gain” approach was suggested by the philosopher Maimonides (Born 1138 in Spain; Died 1204 in Egypt) during the Middle Ages when “Talmudic” scholars attempted to resolve various hypothetical bankruptcy problems.

Faced with the idea of entity equity from the viewpoint of “gain,” it may be suggested that an alternate view in entity equity is dividing up the loss equally. How much is lost collectively by the claimants? The loss $L = Total\ claims - E$. How the losses are “distributed” among the claimants may catch one’s attention in considering if a fair result has occurred.

The analogue of the method of total equality of entity gain suggests that there be a method where each player is given an equal amount of loss. This, however, is not always possible without some players adding their own money to the estate E , and just as giving people more than they claim, asking some claimants to “subsidize” the settlement will be rejected by many. However, it may be possible to give scenarios where this “subsidization” might have some appeal. Just as one can have a constrained equality of gain approach as mentioned above, due to Maimonides, one can also have constrained equality of loss, without subsidization, and this system was also mentioned by Maimonides. To repeat, the idea is to equalize the loss as much as possible among the claimants until the amount E is exhausted and not to enlarge the estate with contributions from the claimants themselves to achieve total equality of loss.

One appealing aspect of the method which tries to equalize the losses of the claimants is that it leads to the standard algebra topic of solving two equations in two unknowns in a natural way.

Consider this problem:

Example 1: Suppose E (remaining assets with which to settle claims) = \$210
Claimant A has verified claims of \$60.
Claimant B has verified claims of \$300.

If we give A , a units, and B , b units we have the following equations:

$$a + b = 210$$

$$60 - a = 300 - b$$

Solving, we find that $b = 225$ and so $a = -15$.

Whoops? What does the -15 mean? The only way to equalize the loss total is for A to add 15 dollars to the estate, so A 's loss will be $60 + 15 = 75$, and B 's loss will be $300 - 225 = 75$ also.

If we are not happy with this approach we can use Maimonides loss, which will mean that we try to make the losses of A and B as equal as possible. Thus, if we give B all of the 210 units of E , thereby making at this stage A having a loss of 60 units and B a 90 unit loss. Without A "subsidizing" the value of E with more units, there is no way to make the losses of both more equal, so the Maimonides loss solution gives A nothing (0 units), and gives B all 210 units.

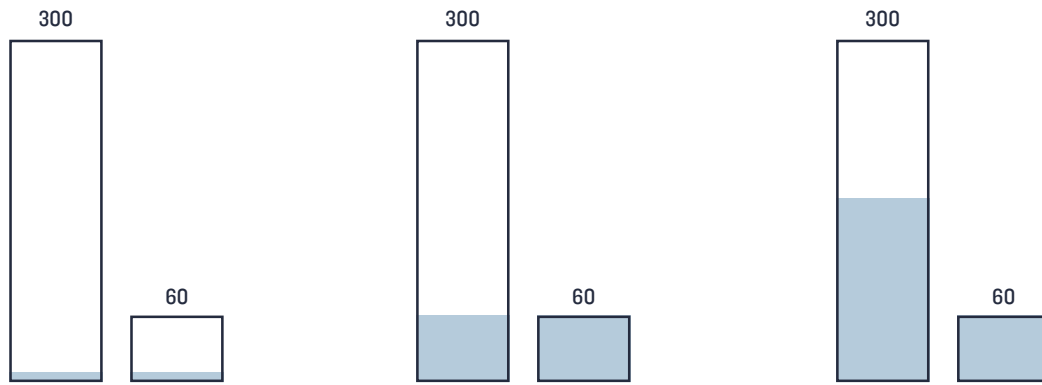
It might seem as if to solve the constrained optimization problems involved in Maimonides gain and loss, one would have to work with equations or inequalities as we did in trying to equalize losses as indicated above. However, there is an elegant "visual" method which vastly simplifies thinking about these two methods of attacking fairness algorithms for bankruptcy problems.

Think of the estate as a colored homogeneous fluid (blue in our diagrams) and the claims as "glass containers" whose heights are the claims. We imagine that we have a system of filling up the containers (bins) that represent the claims with the fluid, with the filling stopping when one reaches the top of a claims container.

The order in which one lists the containers is somewhat a matter of discretion. Here (Figure C.13) we have filled the claims containers equally with a small amount of fluid. We continue to fill the two containers equally until the smallest claim is "topped" off when it reaches the height of the claims container.

At this juncture (Figure C.14) we have filled both containers to the height of 60. Since there are two containers at this height we have used 120 units of the estate thought of as fluid. So we have $210 - 120 = 90$ units of fluid left. This (Figure C.15) allows us to continue adding to the container representing a claim of 300 to the height of 150. We have used the total value of E , since $150 + 60 = 210$. Note the diagrams above are "symbolic" and don't have to be metrically accurate in order to do the arithmetic that solves the problem.

The same containers can be used in one's mind's eye to solve constrained loss equality problems. Defying the laws of gravity one needs to pare down the loss of largest claimant, if possible to the loss of the next largest claimant so as to try to equalize their losses. In this example if we start to "fill" the 300 container starting at its top, we can put 210 units of fluid, all of the estate before reaching the level at the top of the other claimant, whose container has height 60. Thus, to try to equalize the claims, all of the estate E of 210 must be given to claimant B , with 0 given to claimant A .



**FIGURE C.13: SMALL EQUAL AMOUNTS
HAVE BEEN DISTRIBUTED TO THE TWO
CLAIMANTS**

**FIGURE C.14: THE CLAIM OF 60 HAS
BEEN MET**

**FIGURE C.15: THE ESTATE HAS BEEN
AS EQUALLY DISTRIBUTED AS
POSSIBLE**

PROPORTIONAL APPROACHES

Another approach often “invented” by students is: Assign each claimant an amount proportional to the size of its claim.

In fact, the idea of this approach is sometimes triggered by the fact that when discussing how the US Senate treats all states equally, students will raise the idea that in the US House of Representatives states be given a number of seats which is in proportion to their population. Since states can not be assigned a fraction of a seat, the equity problem in designing a fair way to give states seats in the House of Representatives has rather different ramifications than the bankruptcy problem and is known as the apportionment problem. The apportionment problem for the US House of Representatives (there are also other interesting apportionment problems, for example, assigning parties seats in European parliaments in proportion to the vote they get in elections) has the additional Constitutional requirement that each state get at least one seat. For bankruptcy problems, there are methods where a claimant will be given no part of E at all!

In light of the discussion above one could also assign each claimant an amount computed from first distributing the total loss proportionately based on the claims, and then giving the claimants what they deserve from this point of view. Using a bit of algebra it is possible to see that the two seemingly different approaches in fact give each claimant the same amount from E .

There are two other approaches to the proportional way of assigning claimants parts of E . One of these is based on the following idea. Since E is not enough to pay off the claimants, why not invest E until it grows to the total claimed at the current rate of interest and then disperse the amounts necessary to pay off the claimants? However, this would entail having

the claimants perhaps having to wait a long time for their money. So, after computing the amount of time that it takes E to grow to the total claim, compute the present value of those future claims. It is not hard to see that these present values would assign each claimant the proportional solution.

Another way to view the proportional solution would be to say, since we can't pay off the claimants fully, how about give each claimant c units for each unit of claim they have, where c is less than one unit. (This can be thought of as giving each claimant c cents for each dollar of a monetary claim.) It is not hard to see, using some algebra, that this approach, too, yields the proportional solution.

Note that when proportionality is used, no claimant is ever given all of their claim. However, with some of the other fairness methods above, some claimants will get all of their claims but others won't. This bothers some people, but others feel that this is what fairness requires.

Summarizing, we have now looked at 6 different ways of thinking of solving bankruptcy problems:

- Entity equity for gain
- Entity equity for loss
- Maimonides gain (constrained equality of gain)
- Maimonides loss (constrained equality of loss)
- Proportionality of gain
- Proportionality of loss

But there are other ideas!

Suppose that one picks an ordering of the claimants at random, thus not giving any advantage to any claimant, and disperses the amount E as follows.

- Starting with the claimant at the top of the ordering give that claimant all of what is claimed or if the claim exceeds the amount E , just all of E .
- If there is more money left, then
- Repeat i . with the second person on the ordering.

This approach is repeated until all of E is dispersed.

However, while it may seem "evenhanded" to use a random approach, many find this approach unappealing. But, there is a germ of an idea here that does restore fairness in some people's minds. Consider all orders of dispersing the estate E to the claimants, as above, that is, give the money out in the order the claimants appear in a given one of the $(r!)$ orderings, and then take the mean of the amounts that each claimant gets over all of the possible $r!$ orderings (r claimants, $r!$ orderings).

This solution, which appears in various guises in different parts of game theory, is known as the Shapley Value, and it is part of the reason why Lloyd Shapely won a Nobel Prize in Economics in 2012 (shared with the game theorist Alvin Roth).

Like the many appealing and different methods that can be used for deciding an election using ranked ballots, we have seen now a variety of appealing methods for solving bankruptcy problems. When having students model bankruptcy problems, how many claimants is it wise to have in sample instances that students might consider? While realism requires problems with reasonably large number of claimants, and certain aspects of the way the different methods distribute the estate E seem more reasonable when one has many claimants, there is the modeling education idea that one should try simpler cases than the actual problem at hand in order to get insight. For bankruptcy problems there is another reason to look at the two-claimant case that seems very appealing and which has no natural counterpart for bankruptcy problems with more than two claimants.

I will describe a method for solving a two-claimant problem which appears in discussions of bankruptcy problems from the Middle Ages, and which, I, at least would never have thought of in a million years!

Let me illustrate with this example, and here as in the instances above, I will use letters A , B , etc. for the claimants' names and the size of their claims, and the letters a , b , etc. for the amounts that a given method will give to the claimants. As usual, E will stand for the size of the estate, so I will not use examples with more than 4 claimants. Also, I will for convenience list the claims in non-decreasing order of size.

$E = 200$ with A 's claim of 120 and B 's claim of 180

Here is how some scholars attacked this problem in the Middle Ages:

Suppose A goes to the person administering the 200 units of an estate and says: "Look, there are only two claimants, me (A) and B . B is only claiming 180 so at least 20 units of the estate must be mine." This amount, 20, is called A 's uncontested claim against B . For some values of the claims of A and B , B cannot make a similar argument (because A is asking for the whole estate or more) but in this example, B has a similar argument.

Claimant B can go to the administrator dispersing the estate E and say, " A is only asking for 120 so at least 80 units of the estate must be mine." So B 's uncontested claim against A is given by 80 units. Now each of A and B are given their uncontested claims, which add to 100 units. This means 100 units still to be distributed ($200 - 100$). Since the remaining amount (100) is contested by both A and B the administrator splits this equally between A and B . So A gets $20 + 100/2 = 70$ and B gets $80 + 100/2 = 130$. As a check, $70 + 130$ equals the size E of the estate, 200. Note that it is not totally obvious that there will always be additional funds to

distribute after the uncontested claims are met, but there always will be.

This procedure is sometimes called the Talmudic Method, sometimes the contested garment rule, and sometimes concede-and-divide. The use of the term contested garment rule to describe this method stems from problems that are set as two claimants contesting a single garment. But bankruptcy problems are usually set in situations where subdivision of the estate E involves distributing something that is homogeneous and “infinitely divisible.” Strictly speaking money is not infinitely divisible because there is a smallest unit of currency, in America, a penny. However, the value of a garment is destroyed when it is subdivided to give the parts to claimants. Bankruptcy problems are usually stated in terms of distributing something whose value is not destroyed when it is subdivided, as would be the case for a garment, painting, or other kinds of assets, which might be in an estate. Thus, in recent bankruptcy literature the method called “the contested garment rule” is now often called concede-and-divide.

The recent interest in looking at bankruptcy problems was stimulated by a paper by the political scientist Barry O’Neil. The problem stimulated Michael Maschler and Robert Aumann to do some seminal work on the problem. O’Neil called attention to a problem described in the Babylonian Talmud summarized in the table below, which shows solutions for three different bankruptcy problems with three claimants.

The entries in the body of the table represent solutions which correspond to the estate sizes of 100, 200, and 300. The top line looks like equal division while the bottom line looks like proportionality, but the middle row of numbers is puzzling. Aumann and Maschler showed there was one method, now usually called the Talmudic Method, which will explain all three rows!

		CLAIM		
		100	200	300
ESTATE	100	33 1/3	33 1/3	33 1/3
	200	50	75	75
	300	50	100	100

TABLE C.7: THREE BANKRUPTCY PROBLEMS DESCRIBED IN THE BABYLONIAN TALMUD.

What Aumann and Maschler did constitutes a “mathematical detective story,” which is interesting in its own right and can be found in Malkevitch [2005] but here I will just describe the ingenious algorithm that they found for explaining the numbers in Table C.7. The method can be applied to any of the rows of the table, but I will show how it applies to the puzzling second row of the table.

The basic idea is to divide the claims of all the claimants in half, to get “half-claims” for all of the claimants. First, take the first collection of half-claims and use the estate E to settle the claims using the Maimonides gain method. If there is money left over, use it to settle the other collection of half-claims using the Maimonides loss method.

So for an estate of size 200 (Table C.7, second line), A , B , and C ’s “half-claims” are 50, 100, and 150.

Applying Maimonides gain, we try to give A , B , C as equal amounts as possible but not exceeding their claims. So we give A , B , and C 50 each, at which point we have distributed 150 of the 200 units available. Now we continue to treat B and C equally, giving them each 25 additional units, which gives A 50 units, B 75 units, and C 75 units, which exhausts the estate E and so we are done. (Typically, however, one might have enough of value in the estate that after the first half-claims are met using Maimonides gain, one needs to work on the second half-claims using Maimonides loss.)

The method of Aumann and Maschler (let me call it the Talmudic Method) when applied to the case of 3 or more claimants satisfies a remarkable property known as consistency. Suppose one looks at a subset T of at least two claimants but not the full collection of claimants. Suppose that the Talmudic Method assigns T an amount E^* . Then the Talmudic Method, when used to settle the same sized claims but using E^* instead of E , distributes the amount E^* to the members of T with exactly the same amounts as in the original solution.

In particular, if T has exactly 2 members, then the amounts distributed are what would be given by the concede-and-divide (contested garment) method.

For an illustration of the idea of consistency, consider the Talmudic Solution for an estate of 200, the second row of Table C.7. The amount assigned in this case is $A = 50$, $B = 75$, $C = 75$.

To check “consistency” in this example requires that one check three cases:

- $E^* = 125$, for claims of $A = 100$ and $B = 200$
- $E^* = 125$ for claims of $A = 100$ and $C = 300$
- $E^* = 150$ for claims of $B = 200$ and $C = 300$

For the first, A has no uncontested claim against B but B has a 25 unit uncontested claim

again A . So give B 25 units and split the remaining contested claim of 100 equally. A gets 50 and B gets 75, the same amounts as originally.

For the second, A has no uncontested claim against C but C has a 25 unit uncontested claim against A . So give C 25 units and split the remaining 100 equally. A gets 50 and C gets 75, the same amounts as originally.

For the last, neither B nor C has an uncontested claim against the other, so the estate of 150 is equally divided. B gets 75 and C gets 75, as was true for the original settlement.

Consistency seems like a strong condition and it is remarkable that the Talmudic Method, with its balance of attention between gains and losses, satisfies this condition.

Many “fairness axioms” have been proposed that appealing methods to solve bankruptcy problems should obey. For example, if the estate goes from E to which is larger, and the claims stay the same, then no claimant should get a smaller amount when the method is applied using rather than E . If two bankruptcy problems differ only in that one claimant’s claim has gone up, then a reasonable method will not give that claimant less than when the claim was smaller. Researchers on bankruptcy problems have worked towards determining which methods obey which fairness rules, and ideally towards being able to say that if one wants a particular collection of fairness rules to hold, then the only method one can use is some specific method.

In recent years much of the effort in understanding the bankruptcy problem has been carried on by William Thomson, a mathematical economist who teaches in the Economics Department at the University of Rochester. Thomson’s extensive research (and that of his students) has greatly elucidated the remarkably rich collection of methods that solve bankruptcy problems and the fairness axioms these methods obey. Thomson’s [2015] recent survey is a treasure trove of fascinating results and ideas.

Bankruptcy problems are but one of a wide range of fairness issues that can be modeled by prekindergarten-12 students. Others include elections, apportionment, cost allocation, and weighted voting. Remarkably, like the modeling done above, while the theory may involve work well beyond what can be treated in prekindergarten-12, often the algorithms involve little more than arithmetic and algebra! For additional readings on these and other fairness problems see the reference list on p. 212.

DRIVING FOR GAS

INTRODUCTION

This investigation is suitable for use as a first modeling activity with students in algebra or precalculus level classes. It is an especially good problem for students just getting their

driver's licenses since they have a vested interest in its solutions. In addition, relevant data are available on the internet through sites such as "Gas Buddy." Because of their familiarity with the context, students can readily identify relevant quantities of interest and reasonable simplifying assumptions without teacher assistance, and productive analysis may be carried out using nothing more than rational expressions and linear functions.

TEACHING PRINCIPLE

Modeling is about decisions — recognizing when a decision is necessary, making a decision, and revisiting decisions. Helping students develop these skills is a primary goal in teaching modeling. Whenever possible, let students "drive" the discussion, helping them especially with the "recognize" part of their own decision-making.

General modeling principles

- It is usually easier to develop useful models by starting with a simplified version of a situation than with one that is closer to reality. The first model is rarely the final model.
- Pay attention to what you "want." If you need a number, make up a value, but note what you did. That number may become a variable later.
- Be conscious of decisions/assumptions.
- Ask, "What if?" What would happen if (pick a number or assumption) changed?
- Ask, "What question are we trying to answer? How can I 'measure' that?"

PROBLEM STATEMENT

Most drivers have a "usual" region in which they do most of their driving. However, gas prices may vary widely so that gas may be substantially cheaper somewhere other than within that usual region. Would it be more economical to go to a station outside the usual region to buy gas? Thus, the general question we wish to address is, "How might we determine which gas station is the most cost-efficient?"

ASKING THE QUESTION

First, we need a more clearly defined question to consider. Since the answer to the main question above will vary depending upon location and vehicle, a more "local" form of the problem might help students get started. Students with experience modeling should be able to do this refinement on their own. But, for students new to modeling, the teacher will need to help them formulate the problem in a more tractable and clean form and may choose to begin with the formulation below rather than the one above. For example,

You drive to school every day. On the route you take from home to school, there are several gas stations. Unfortunately, the prices on your route are always high. A friend tells you she buys her gas at a station several miles off your normal route where the prices are cheaper. Would it be more economical for you to drive the extra distance for the less expensive gas than to purchase gas along your route?

Even this localized version of the problem is still open-ended. From this starting point, what questions should students be asking themselves? How should the teacher support them as they create a model to answer the question?

CREATE A SPECIFIC EXAMPLE

We always encourage our students to begin by creating a small, very specific version of the problem for themselves. Asking students to put themselves into the scenario and try to answer the main question for themselves is a good way to get them started. Using special cases helps students clarify the problem and determine important variables, and it often suggests approaches that could be useful in the more general setting. Having them apply the situation to themselves permits students to use what they know about their own route as the “facts” of the problem.

The teacher might help students realize, too, that their “personal” version of the problem does not have to be “the truth.” Prices and distances can be chosen for convenient calculations instead, allowing the structure of the problem to be more visible. In addition, teachers should have students explicitly note what quantities they have defined along the way. These quantities may be revisited later in the modeling process to become variables in a more general model.

One example of a “personalized” problem statement is:

Suppose there is a station on your normal route that sells gas for \$3.00 a gallon. A station 5 miles off your route sells gas for \$2.85 a gallon. Should you travel the extra distance to buy gas at that station?

Students having difficulty might also be encouraged to draw a diagram to illustrate the problem.

Note: A general modeling principle is that it is usually easier to develop useful models by starting with a simplified version of a situation rather than with one that is closer to reality.



FIGURE C.16: DIAGRAM ILLUSTRATING THE SIMPLE PROBLEM

The first model is rarely the final model. Thus this diagram shows the trip to station B as perpendicular to the main route, where the real-world configuration might be different.

Based on the main question of the problem, students should see that they need to compare the total costs of buying gas at the two stations. As they start those calculations they will discover that another necessary quantity is missing—the amount of gas to be purchased. As before, choosing a specific value allows the student to move forward and provides another variable for a general model. For simplicity, let's assume that you always fill up the tank when the tank has 4 gallons remaining (on new cars, a light often comes on to indicate this level). Suppose your car has a 14 gallon tank.

At this point, students can compute the costs of buying gas at the two stations. However, the problem statement above still needs refining before it really represents the question of interest. To help students see the need for further clarification, ask “Why would we not drive the five miles to buy cheaper gas?”

If we changed the problem so that the cheap gas is 50 or 100 miles away, students quickly see that they would use up any money saved in purchasing the less expensive gas by driving to the station. This tells us that the amount of gas consumed in traveling to the station is important since it does not help in our getting back and forth between home and school. Free gas at the fringe of our driving range gets us no useful miles.

Since the gas consumed depends on the efficiency in miles per gallon (mpg) of their car and the distance driven, the car's mpg and distance would be important factors. It also suggests that the answer to the question might be different for two students who drive different cars. The student driving an SUV getting 10 mpg might be less likely to drive the 10 miles (5 miles out to the station and 5 miles back to your route) than a student driving a small car getting 30 mpg.

Note: This discussion points out the need for an assumption that only money, and not time, will be used in this initial cost model.

This leads to a more refined statement of the problem:

Station A is on your normal route and sells gas for \$3.00 a gallon while Station B, which is 5 miles off your normal route, sells gas for \$2.85 a gallon. Your car gets 30 mpg, and your friend's car gets only 10 mpg. Should either of you drive to Station B for gas?

Now we have a problem we can do some interesting mathematics on!

Note: Assuming constant values for prices and distances is consistent with reality. The

implicit assumption here that mpg is a constant is less “correct” but is exactly the right kind of assumption to make in formulating a first model. Be sure students notice this decision.

SOLVING THE SIMPLE PROBLEM

As noted above, both the trip to and from the gas station represent distance outside of the routine driving region, so these are not “useful” miles in the commute. Taking that fact into account, a typical student model for the problem looks like:

$$\text{total cost} = \text{cost of purchased gas} + \text{cost of additional trip for gas.}$$

(See below for alternate formulations students might produce.)

Students should use their model to answer the refined version of the problem. The discussion below reflects how students might proceed and can be used to help “stuck” groups make progress.

Let’s first consider your car. For Station A, there is no cost to drive to the station since it is on your normal route and you drive by it every day anyway. (Note: This represents an assumption making the problem easier to think about. We can modify this later if we choose.) You just pull in and pay \$3.00/gallon to fill up. As noted above, how much you spend depends on how many gallons you normally purchase when you “fill up” the tank. So, the size of your gas tank is an important variable.

At Station A, you put in 10 gallons of gas for a total cost of \$30. For Station B, you must account for driving 10 miles out of your way. So, you must buy 10 gallons of gas for the tank and an additional amount to replenish what was used in driving. You used

$$\frac{10 \text{ m}}{30 \text{ mpg}} = \frac{1}{3}$$

of a gallon of gas getting to and from the station, so $28.50 + (1/3) \cdot 2.85 = 29.45$. You can save 55 cents by driving to the more distant station.

Your friend’s cost is the same \$30 at Station A, but their cost for Station B becomes $28.50 + (1) \cdot 2.85 = 31.35$ since they used a full gallon in travel. It is not a good deal for your friend to make the trip.

ALTERNATE FORMULATIONS (“SIMPLER PROBLEM”)

In working through the arithmetic version of the problem, students can develop a variety of simple models. Some consider the cost per gallon of the gas rather than the cost per fill-up. They note that not all of the gas purchased is “useable”, since some of it is burned in travel. They would argue thusly:

For Station A, there is no cost to drive to the station since it is on your normal route and you drive by it every day anyway. You just pull in and pay \$3.00/gallon to fill up. You put in 10 gallons of gas for a total cost of \$30 and all the gas you purchased is “usable”, so you paid \$3.00 per usable gallon. For Station B, you must drive 10 miles out of your way. You buy 10 gallons of gas for \$28.50, but not all of it is “useable gas” since you will use

$$\frac{10 \text{ m}}{30 \text{ mpg}} = \frac{1}{3}$$

of a gallon of gas getting to and from the station. So you have paid \$28.50 for 9.67 gallons of useable gas, or

$$\frac{28.50}{9.67 \text{ gallons}} = \$2.95 \text{ per gallon of useable gas}$$

It is certainly to your financial advantage to drive to Station B. What about your friend?

Your friend will also purchase 10 gallons of usable gas for \$30.00 at Station A. Her price per gallon of useable gas is \$3.00. However, at 10 mpg, she will use a full gallon on the drive to Station B. Consequently, she will pay \$28.50 for 9 gallons of useable gas. That translates to \$3.17 per gallon of useable gas, which is not such a good deal.

You should drive your car to Station B for the less expensive gas, but your friend should stick with Station A.

ASSESS THE MODEL AND REVISE

As it stands, the initial model does a reasonable job of answering the question, but only for one or two specific individuals. Students are now in position to see how results differ if a prior decision changes. One possible choice is tank size.

Considering the size of the gas tank Students will know that most big cars with low mpg have larger gas tanks. Does this make a difference? Suppose your friend’s car has a gas tank that allowed a 25 gallon purchase instead on only 10?

If students compare the cost of filling the tank, at Station A it is now \$75 while at Station B, it is $(25) \cdot 2.85 + (1) \cdot 2.85 = 74.10$ dollars. Your friend could now save nearly a dollar by making the drive.

If students consider the per gallon cost of usable gas, then having the larger tank does not affect the cost at Station A; it’s still \$3.00 per useable gallon. At Station B, however, she purchases 25 gallons at \$2.85 per gallon for a total cost of \$71.25. Since a gallon was used in driving to station B, she buys only 24 gallons of usable gas, which is

$$\frac{71.25}{24 \text{ gallons}} = \$2.95 \text{ per useable gallon}$$

That's a slightly better deal, saving 3 cents per gallon of usable gas. Whether this savings is worth the time and effort is a different question. We do see that the size of the gas tank (the amount of gas we can purchase at one time) can be an important component of the problem.

TEACHER INTERLUDE (FORMATIVE ASSESSMENT)

If we are working on this problem in a large class setting, we might want to inject an example at this point to check the students' understanding.

Example Problem There are 4 gas stations that sell gas for different prices. Station A is on your normal route and sells gas for \$3.00/gallon. Station B is 5 miles from your route and sells gas for \$2.85/gallon. Station C is 10 miles from your route and sells gas for \$2.70/gallon, while Station D is 2 miles off your route and sells gas for \$2.90/gallon.

Consider three cars. Car 1 gets 10 mpg and has a 12 gallon tank. Car 2 gets 10 mpg and has a 27 gallon tank. Car 3 gets 35 mpg and has a 12 gallon tank. From which gas station should the driver of each car purchase their gas?

By working through simplified versions of the problem with specific numerical values, arithmetic is the important tool rather than algebra. The focus becomes the structure—determining an appropriate method of comparison and the arithmetic of making that comparison. One of the most common difficulties students have in modeling is trying to generalize too soon, thus making the problem too hard before they have seen the underlying structure. Playing around with specific cases of the problem, particularly by considering extreme cases as illustrated here, is very important, and the time taken to play with the ideas should not be cut short.

AN ALGEBRAIC SOLUTION: ASSESS THE MODEL AND REVISE

As it stands, the simple model “works,” but only in one-at-a-time calculations. Each individual driver and station combination must be checked separately. But now students have a good idea about how the problem works with specific values and can write an algebraic representation of their initial model. This representation will allow them to find a general solution.

The important quantities have been identified. Students know that the prices, total distance driven, the mpg of the car, and tank size (volume of gas purchased) are all important. At this point students must employ a generally applicable principle of modeling—either assign a quantity a fixed value (adding an assumption, the method used in the simple model above) or make it a variable to be carried through the work. The remaining discussion takes this

second option, using letters to denote the important quantities.

- Let p represent the price per gallon at the station along our route and P the price per gallon at the station we are considering.
- Let D represent the distance in miles from the normal route that must be driven to get to the gas station (so a round trip is $2D$ miles).
- Let M represent the miles per gallon of the vehicle.
- Let T represent the number of gallons of gas we purchase when we buy gas (in our model, four gallons less than the tank size).

ASSUMPTIONS IN THE MODEL

The problem statement implies that driving takes place in a well-defined “routine” region and only stations outside that region merit special attention. As an initial assumption, the trip to and from the “outside” gas station is viewed as a round trip from and to a given point in the routine region, as embodied in the diagram above.

We also assume in our model that gas mileage, M , is constant. Students know that it varies depending upon the driving conditions and speed of travel. Similarly, the amount of gas to be purchased has been assumed to be the same for each fill-up. In a more refined model, we might revisit these assumptions.

Basic model 1: cost of filling the tank Using the formulation, *total cost = cost of purchased gas + cost of additional trip for gas*, the question becomes: minimize C , where

$$C = T \cdot P + \frac{2D}{M} \cdot P$$

and all quantities are non-negative (permitting $D = 0$ for stations in the routine region).

Basic model 2: cost per gallon of usable gas We have seen that the price per usable gallon for the station along your route is just the price per gallon, p , since all of the gas purchased is usable. We are therefore always comparing our cost per usable gallon, in terms of P , to p .

To compute the price per useable gallon for the other stations, we compute the cost of the tank of gas, $T \cdot P$, and the number of usable gallons

$$T - \frac{2D}{M}$$

The cost per usable gallon is the ratio of these two values

$$\frac{T \cdot P}{T - \frac{2D}{M}} = \frac{T \cdot P \cdot M}{M \cdot T - 2D}$$

Now we have a formula that drivers can use to make their decisions. We should buy gas at the distant station if

$$\frac{T \cdot P \cdot M}{M \cdot T - 2D} < p$$

As before, most of these letters represent quantities that do not vary.

Basic model 3: cost per mile of travel One way to rethink the initial approach is to consider what is actually purchased in the transaction. Reflection leads to the realization that the driver is really buying “miles” rather than gas. Since miles “used up” going to and from the gas station are not miles available in the routine driving region, this gives a new model for the question, minimize C^* , where

$$C^* = \frac{T \cdot P}{M \cdot T - 2D}$$

Again, all variables are non-negative, but D is 0 for stations in the routine region. (Note that in this formulation, C^* represents cost per mile rather than total cost. As a result, cheap gas at great distances now appears less competitive than in Basic Model 1 since the useful miles decrease with distance. This model is mathematically equivalent to Basic Model 2 but reflects a different approach in the thinking.)

USING THE MODELS

Note that each of the above models involves three variables—the cost we wish to minimize, price, and distance, and two parameters—volume of gas purchased and mpg of the car. Students may not have encountered this many quantities in their previous work, so may need assistance in using their models.

Perhaps the simplest way to apply an algebraic model is to use the model as a direct formula for computing the measure of cost that is of interest. Each candidate station would be evaluated separately by substituting the relevant information for that station into the formula.

Students can use that same idea more efficiently by creating a spreadsheet to tabulate computed costs C^* for a range of values of price P and distance D . The example shown uses cost per mile purchased model (model 3).

For students new to modeling, this would be a reasonable stopping point for this investigation. Summary discussion should continue to focus on noting the decisions that were necessary in getting to this point.

For students who are comfortable working with linear functions a more geometric

DISTANCE	PRICE	COST
5	2.25	0.096626
5	2.27	0.097485
5	2.29	0.098344
5	2.31	0.099202
5	2.33	0.100061
10	2.25	0.099684
10	2.27	0.10057
10	2.29	0.101456
10	2.31	0.102342
10	2.33	0.103228
15	2.25	0.102941
15	2.27	0.103856
15	2.29	0.104771
15	2.31	0.105686
15	2.33	0.106601

TABLE C.8: COST PER MILE PURCHASED, WHERE THE FUEL TANK CAPACITY IS 14 GALLONS AND THE CAR GETS 24 MILES PER GALLON.

approach to using the model is possible, though a little prompting from the teacher may be necessary at the start. Since the real interest is in how things behave as different stations are considered, the variables that describe stations, namely P and D , can become the focus. The “trick” is to again think of cost as a constant and rewrite D in terms of P . Again using Model 3, and remembering to treat M , T , and C^* as constants reveals a linear relation between distance and price:

$$D = \frac{TMC^* - TP}{2C^*} \quad \text{or} \quad D = \frac{TM}{2} - \frac{TP}{2C^*}$$

When considered in the P - D plane, both the slope and D -intercept of the line are determined completely by the tank size and mileage of the particular vehicle and the value of C^* . Thus,

all stations that lead to the same overall cost measure are described by points on a single line. In addition, changing C^* just changes the slope,

$$\frac{T}{2C^*}$$

so the goal of minimizing C^* becomes choosing the best line from an infinite family of “radial” lines through the common vertical intercept

$$\frac{TM}{2}$$

Put into action, comparing stations reduces to comparing the lines upon which the competing stations lie. That is, plot the price and distance information for the competing stations in the P - D plane, then “swing” a line clockwise around the “pivot point” on the vertical axis to the “last” point for a station (this is reminiscent of the Which Computer? problem). This means that the determination of the best station can be made entirely geometrically after calculating only the vertical intercept.

Note that the vertical intercept,

$$\frac{TM}{2}$$

has a nice interpretation—it’s half the range of the vehicle. So adjusting the model to apply to another vehicle involves knowing that new range.

Of course, other representations of the models are possible, including using a 3-D graphing utility to view the graph of C directly as a function of P and D .

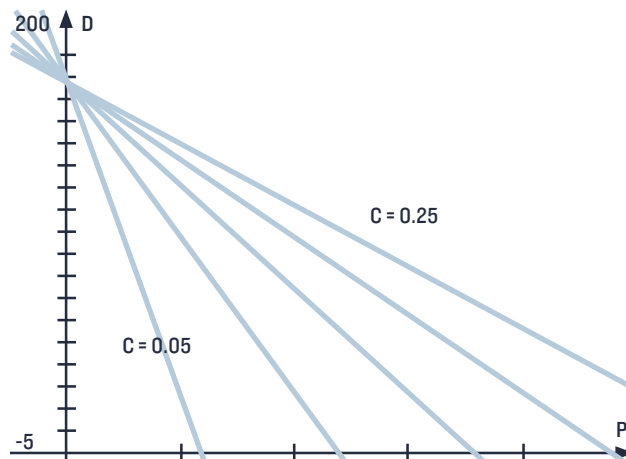


FIGURE C.17: TYPICAL SWINGING LINE VISUALIZATION

CONCLUSIONS

The primary goal of the Driving for Gas scenario is to emphasize the importance of encouraging students to take the initial time to play with small problems using simple numbers. By spending time with these special cases, students can develop a feel for the problem, can see what is important and what can be left out, and can use their work to develop a more general form of the model. Our experience with students new to modeling is that they want to jump to the algebra before they have really understood the structure of the problem. By not spending the time early to play with the situation, they often miss important ideas that could greatly simplify and improve their work.

D

APPENDIX D: ASSESSMENT TOOLS

This appendix contains examples of assessment tools, including checklists and rubrics that you might consider using when teaching with modeling.

In the first section we discuss a formative assessment plan for use with active modelers engaged in various stages of the modeling process. The remaining sections include examples, and brief discussion of suggested usage, of rubrics and other tools designed primarily for assessing overall output from mathematical modeling activity.

SAMPLE QUESTIONS FOR ASSESSMENT OF THE MODELING PROCESS

ENGAGING STUDENT MODELERS

The modeling process consists of inherently interconnected components that are used in an iterative fashion, yet it also has individual identifiable components that modelers necessarily encounter when working on a project. In this section we discuss a collection of questions designed for formative assessment of students engaged in mathematical modeling. The following subsections consist of questions, organized by their connection with a component of the modeling process, which you may consider presenting to students in order to assess their approach to mathematical modeling focused problem solving as well as to help them further their progress as independent modelers. Preceding each subsection is a brief discussion of timing and usage of the stated questions. We do not suggest presenting all questions in a subsection to all students; but we do recommend engaging students in one or more question associated with each component regardless of their ability to demonstrate modeling prowess. Also note that due to the iterative nature of modeling, it is quite possible that the order you ask questions is different from the way they are listed below.

Table D.1, at the end of this section, restates core questions associated with each modeling component and identifies modeling related vocabulary.

DEFINING THE PROBLEM

Soon after initiating a modeling activity, students may have concerns about how to get started or alternatively may have multiple ideas about how to proceed. The following questions, many of which can be asked soon after brainstorming has started, may help modelers develop focus and move forward.

- Describe the problem that your team has been asked to solve. What information do you need in order to solve this problem?
- What does a solution to your problem “look like”? Is it possible that your solution will have more than one reasonable answer? Why?
- What is the specific problem your model is going to solve? How can you complete this sentence “Our model will tell you _____”

MODEL BUILDING – DEFINING VARIABLES AND MAKING ASSUMPTIONS

Brainstorming often leads to the development of numerous valuable ideas one can use to create a model; however students may also find the process overwhelming. For example, a student may determine that they need certain information to solve their problem, yet they are unable to find a value to use in their model. A situation such as this can be approached in a variety of ways; a number of which can be addressed by student responses to these questions.

- Of the factors you have identified as being important to the problem, which values change and which stay the same?
- What assumptions do you need to make in order to find a solution? What prompted you to make these assumptions?
- What would cause you to change an assumption?
- What are the primary factors that you have identified as being important to the problem? How do you plan to incorporate these values into your model?
- Where did you find numbers (or data) to use in your model?
- When researching, did you find more than one value for a factor? How will you determine which value(s) to use in your model?

GETTING A SOLUTION

Creating a mathematical model from “scratch” can be a daunting endeavor, especially for students new to the process. A good starting point for model development builds directly on work that students may have engaged in earlier; but requires an extra step (or two) forward. In particular, encouraging students to identify and reflect on methods to organize the data they have collected or combine values in order to return a value that makes sense in relation to the question under investigation can lead to a meaningful solution. As a result, a number of the questions presented in this subsection are similar (or nearly identical) to those found in other subsections. However, the context in which these questions are presented to

students helps motivate development of mathematical models:

- Describe the specific problem your model needs to solve. What are the units associated with the solution(s)? (If appropriate) How does your solution imply the need for quantification?
- Of the factors you have identified as being important to the problem, how can they be combined in a way that makes “real-world” sense?
- How can you organize your data to share with an individual unfamiliar with your project? (For example, ordering, graphics, statistics, etc.)
- What mathematical techniques have you used to analyze data thus far?
- Describe the mathematics used in (this portion of) your model.
- How do you plan to communicate your final results?

ANALYSIS AND MODEL ASSESSMENT

The value of a model is determined by its ability to provide reasonable (as defined by the user or audience) solutions to a given problem. Soon after a mathematical model has been developed – or when investigating the usefulness of an outside model – it is advisable to analyze the model. In short, responsible math modelers will verify the applicability and understand the performance limits of their model before sharing results with others.

- How does your model work? What type of values can be input into the model? What does the output “look like”?
- How do you plan to communicate your results? Do you think a graphic might help people understand your information, model, and results? What type of graphic(s) are most appropriate?
- How does the model answer the question you were asked? Does your answer make sense? (For example, are the units appropriate?)
- How do you know your model “works”? That is, do the results make sense in the context of the question you are solving?
- When does your model work? Are there instances in which you need to be careful because it might not?
- How does your model react to changes in input values?
- What improvements would you make to your model? For example, what would you do differently if you had more time? Better data? Something else?

REPORTING RESULTS

In order for a mathematical solution (or solutions) to be of value they need to be understood in the context of the original problem and by an audience interested in learning the result. The final result may be reported in a variety of ways. For example, a written report or oral presentations are common reporting formats. Furthermore, the completed report is commonly assessed in a summative manner (see suggested rubrics later in this appendix).

The questions presented in this subsection encourage student modelers to be specific and thoughtful in sharing their work with others.

- Explain the process your team used to develop a solution.
- Explain the mathematics used to develop your team's solution.
- Who are you sharing your results with? Who is the audience for your report?
- (If appropriate) How did each of your teammates participate in the modeling process? What did you learn from the other members of your team?
- What are three (to five) things you want anyone reading/hearing your report to understand about your model? With your audience in mind, how can you share your results in a clear and concise way?
- What are three (to five) things you want anyone reading/hearing your report to understand about your solution? With your audience in mind, how can you share your results in a clear and concise way?

AFTER THIS MODEL, BEFORE THE NEXT ONE

When students engage in a full modeling activity, it's advisable to take time to reflect on the experience. A few additional questions may provide students with valuable perspectives that will allow for continued, and improved, success with modeling in the future.

- Did you need to revise your model at any point during the activity? If so, why? How did you fix the model?
- Can you identify a math idea that was key to your ability to develop a model?
- How did your modeling strategy change throughout the work period?
- What advice would you give to a classmate (or yourself) prior to developing a mathematical model?
- Given the chance to do this activity again, what would you do? Would you use the same approach or would you alter your plan?
- If given more time, what would you (or your team) do to improve your model or results?
- What was the most surprising (or unexpected) aspect of this project?

MODELING COMPONENT	QUESTIONS ABOUT YOUR MODEL AND HOW YOU MADE IT	MODELING-RELATED VOCABULARY TO BUILD
DEFINING THE PROBLEM	What is the big problem that you have been asked to solve? It might have more than one possible answer.	open-ended problem, constraints
DEFINING THE PROBLEM	What is the specific problem your model is going to solve? (My model will tell you...)	specific, focus
MAKING ASSUMPTIONS	What ideas did you think about that you decided not to try?	eliminate, prioritize
MAKING ASSUMPTIONS	What have you assumed in order to solve the problem? Why did you make these choices?	assumption/assumed
DEFINING VARIABLES	What quantities are important? Which ones change and which ones stay the same?	variable
DEFINING VARIABLES	Where did you find the numbers that you used in your model?	resources, citations
GETTING A SOLUTION	What pictures, diagrams or graphs might help people understand your information, model, and results?	diagram, graph, labels
GETTING A SOLUTION	What mathematical ideas did you use to describe the situation and solve your problem?	situation
ANALYSIS AND MODEL ASSESSMENT	How do you know that your calculations are correct? Did you remember to use units (like dollars or inches?)	calculation, unit
ANALYSIS AND MODEL ASSESSMENT	When does your model work? When do you need to be careful because it might not?	limitations
ANALYSIS AND MODEL ASSESSMENT	How do you know you have a good/useful model? Why does your model make sense?	testing, validation
ANALYSIS AND MODEL ASSESSMENT	If you were going to make your model better, what would you do?	improvement, iteration
REPORTING RESULTS	Explain your mathematical model in words and math.	testing, validation
REPORTING RESULTS	How did each of your teammates help?	model
REPORTING RESULTS	What did you learn from each other member of your team?	collaborate
REPORTING RESULTS	What are the 5 most important things for your audience/client to understand about your model and/or solution?	client, audience

TABLE D.1: MODELING ASSESSMENT RUBRIC (ADAPTED FROM LEVY, IMMERSION).

FEATURES AND USE

The output resulting from a mathematical model varies greatly depending on the preferences of the instructor and, in the real world, on the demands and interests of the client or audience. Moreover, the introduction of a non-traditional (in most cases) mathematical assignment further motivates the need to share modeling project expectations with students.

A mathematical modeling project checklist can be a useful tool for students to fill out prior to completion of an assignment. Additionally, instructors may use a checklist as a summative assessment tool. In this section we provide two examples of checklists that are currently used in mathematical modeling classes. Both checklists can be modified to fit the needs of a specific class or modeling project.

CHECKLIST I

The first checklist, Table D.2, isolates five general items for assessment in a mathematical modeling project. Notably, it provides students with some “common issues” the instructor has identified when reading math modeling based written reports.

CHECKLIST II

The second checklist, Table D.3, similarly provides a collection of categories that need to be included in a mathematical modeling submission. In contrast to Checklist I, this document identifies the need for students to address information “Specific to this project”. Students working on the project associated with this document were asked to investigate the growth of feral cat populations. This collection of four questions can be adjusted for use with alternatives modeling projects.

PROJECT FEEDBACK

- Provided a write-up
- Demonstrated understanding of the premise of the problem: explicit (writing)/ implicit (math)
- Demonstrated proper use of Separation of Variables
- Complete mathematical analysis presented and used to address problem
- Organization (spelling, grammar, format, style, mathematical notation)

GRADE _____

RESUBMIT FOR GRADE

If resubmit for grade is circled, then there are critical errors in either your mathematics or the writing. It might be a good idea to see me before trying again on your own, so I can help you understand what is required.

COMMON ISSUES

Figures & Tables

- Refer to all figures and tables by numbers in the main body of the text
- Figure numbers and captions are required at the bottom of each figure
- Table numbers and titles are required at the top of each table

Tense and Person

- Avoid first person singular (I, me, my)
- Make sure you are consistent in tense throughout
- Passive voice is acceptable in technical writing!

Appendices

- If you use any appendices, be sure to reference it in the main text (otherwise the reader may never know all the good things you've hidden back there!)
- Appendices should either be lettered (A, B, C) or numbered with Roman numerals (I,II,III)

Flow

- There should be a narrative arc—be sure that all ideas are properly introduced, much like you'd need to introduce a character in a story
- Although a non-mathematician might get lost in a couple of the details, anyone should be able to read your paper and get a sense of what you did and why it was important

TABLE D.2: GENERAL ITEMS FOR ASSESSMENT IN A MATHEMATICAL MODELING PROJECT.

MODELING PROJECT CHECKLIST

- Separate title page with names, ID numbers, professor, section, staple
- An introduction which clearly states the problem to be solved
- A paper with distinct sections and concise and clear assumptions, description of model parameters and variable, solution process, summary and conclusions (does your answer make sense and why? What are the strengths and weaknesses of your approach?)
- Use equation editor or hand write equations on their own

ex: $\sqrt{x^2+y^2}$ is bad
ex: $\sqrt{x^2+y^2}$ is good
- Derivations and computations are clear logical and easy to follow?
- A clear description of the variables and diagrams/tables properly labeled with correct units.
- Give acknowledgment where it is due (this included help from people).
References stated.
- Answered all questions being asked, including discussion questions?
- All work is shown? Hand calculations attached and easily referenced?
- Spelling, grammar, and punctuation correct? Is the mathematics correct?

SPECIFIC TO THIS PROJECT

- Proposed a model to predict cat population
- Proposed an intervention strategy
- Used both models to make some future predictions and assessed the quality of the solutions
- Included all parts of the final report guidelines

TABLE D.3: A COLLECTION OF CATEGORIES TO BE INCLUDED IN A MATHEMATICAL MODELING SUBMISSION.

FEATURES AND USE

Mathematical models rarely produce identical results or unique numerical solutions. As a result, formative assessment of modeling projects focus less on determination of a specific answer and more on the appropriateness of the mathematics used to develop a solution and the ability of the modeler to communicate results effectively. The following rubrics are designed to articulate expectations and standards for success on a modeling project. Thus, the rubrics are used by instructors to assess student modeling work and also by students as peer assessment tools. We suggest familiarizing students with the class modeling rubric as early as possible by inviting them to “grade” a third party modeling result. For example, government entities regularly issue modeling based reports; creating an assignment that requires students to use a rubric to assess the executive summary of such a report helps students self-identify what is expected when they communicate mathematical solutions.

RUBRIC FOR WRITTEN WORK

A generic rubric used to assess stand-alone executive summaries can be found in Table D.4. It is significantly more detailed than the score sheets, because of the attention to detail expected in papers that have two pages or less to provide the results associated with a full modeling project. This rubric can be modified for use with “full” multi-page mathematical modeling reports by emphasizing aspects associated with the longer document. In particular, it is recommended that references be identified and that additional detail and attention is paid to analysis and sharing of results.

DEFINE THE MODELING PROBLEM (3 POINTS TOTAL)			
IDEAL	SATISFACTORY	NEEDS IMPROVEMENT	INCOMPLETE
(3 points) Concise problem statement that indicates exactly what the output of the model will be and, if appropriate, identifies the audience and/or perspective of the modeler. Statement is presented early in the paper.	(2 points) Problem statement is easily identifiable but not precise or consistent with other statements in paper.	(1 point) Problem statement is difficult to understand or is buried in the text.	(0 points) No problem statement is given.

BUILDING THE MODEL: MAKE ASSUMPTIONS AND ACKNOWLEDGE LIMITATIONS (3 POINTS TOTAL)			
IDEAL	SATISFACTORY	NEEDS IMPROVEMENT	INCOMPLETE
(3 points) Primary assumptions used to develop the model are clearly identified, easy-to-read and well justified. Limitations due to simplification are stated when appropriate.	(2 points) Primary assumptions are noted; justification or readability is lacking.	(1 point) Assumptions and justification exist, but are difficult to identify in the text.	(0 points) No assumptions –or justification for lack of assumptions is provided.

BUILDING THE MODEL: DEFINE VARIABLES AND IDENTIFY PARAMETER (3 POINTS TOTAL)			
IDEAL	SATISFACTORY	NEEDS IMPROVEMENT	INCOMPLETE
(3 points) Notes and rationalizes the need for the primary factors that influence the phenomena being modeled in a readable format; proper units are specified.	(2 points) Important parameters and variables are listed properly but without sufficient explanation.	(1 point) Variables/parameters are wither barely mentioned or hard for the reader to identify in the text.	(0 points) No variables or parameters are identified.

SOLUTION: MODEL USES MEANINGFUL MATHEMATICS (4 POINTS TOTAL)			
IDEAL	SATISFACTORY	NEEDS IMPROVEMENT	INCOMPLETE
(4 points) Provides a readable glimpse into the mathematical method(s) used to solve the problem. Plausible approach and outcome is presented.	(3 or 2 points) Mathematical approach is stated, but aspects of the method(s) are inconsistent, difficult to understand or incomplete.	(1 point) Model is stated and/or contains fixable mathematical errors.	(0 points) Model is not presented or contains significant errors.

SOLUTION: RESULTS ARE ACCESSIBLE TO THE AUDIENCE (4 POINTS TOTAL)			
IDEAL	SATISFACTORY	NEEDS IMPROVEMENT	INCOMPLETE
(4 points) Clearly presents a solution that is consistent with the original problem statement. If appropriate, a useful visual aid/graphic is included.	(3 or 2 points) Answer is stated, but aspects of the solution(s) are inconsistent, difficult to understand or incomplete (e.g. fail to identify units of measure).	(1 point) Answer is given without contextual background (i.e. appropriate graphics, proper units, etc.).	(0 points) Solution is not provided.

ANALYSIS & ASSESSMENT OF MODEL (3 POINTS TOTAL)			
IDEAL	SATISFACTORY	NEEDS IMPROVEMENT	INCOMPLETE
(3 points) The viability and reliability of the math modeling solution are addressed. For example, how sensitive is the model to changes in parameter values or altered assumptions? How does it compare to other solutions or historical data?	(2 points) Addressed, but the analysis is lacking proper dimensionality. For example, obvious consequences of the stated outcome are ignored or well-known comparisons are disregarded.	(1 point) Some analysis is provided but without any sense of perspective.	(0 points) No analysis or assessment of model is included in the write-up. Incorrect mathematics used in analysis.

WRITING STYLE & ORGANIZATION (5 POINTS TOTAL)			
IDEAL	SATISFACTORY	NEEDS IMPROVEMENT	INCOMPLETE
(5 or 4 points) Correct spelling and grammar is used throughout. Paper is well formatted and enjoyable to read. Visual aids (if appropriate) are well chosen and easy to interpret.	(3 or 2 points) Multiple spelling, formatting or grammatical errors. Visual aids are missing key readability features or do not clearly connect to the solution.	(1 point) Significant disregard for common spelling, grammatical and mathematical rules.	(0 points) Complete disregard for common spelling, grammatical and mathematical rules.

TABLE D.4: A GENERIC RUBRIC USED TO ASSESS STAND-ALONE EXECUTIVE SUMMARIES.

PRESENTATION RUBRIC

The following score sheet, Table D.5, simultaneously serves as a rubric for developing an oral presentation of mathematical modeling results, a peer assessment tool for students, and an assessment tool for instructors. The supplied document is the version provided to students when they prepare to assess peer presentations. One suggested use is to have individuals or teams assess each other so that following presentations they can immediately provide feedback to each other.

MATH MODELING PRESENTATION SCORE SHEET

Presentation made by team:

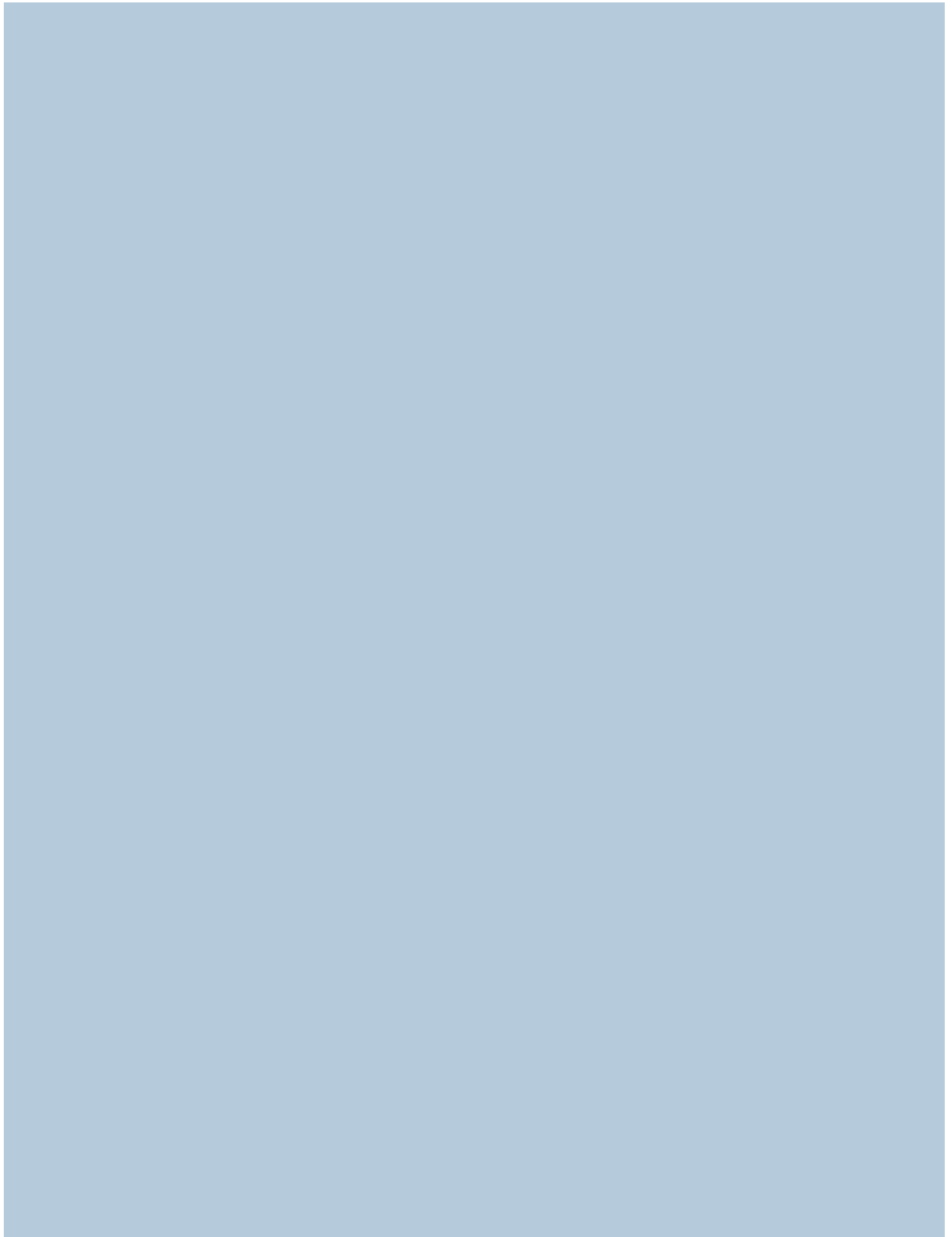
Please select a value (1-5) reflecting the extent to which you agree with the given statement.

	STRONGLY DISAGREE	DISAGREE	NEUTRAL	AGREE	STRONGLY AGREE
I understood the presenting team's interpretation of the question.	1	2	3	4	5
All stated assumptions were adequately justified.	1	2	3	4	5
The model's strengths and weaknesses were addressed.	1	2	3	4	5
Appropriate mathematics was used to create the model.	1	2	3	4	5
A final solution was clearly presented.	1	2	3	4	5
The mathematical model produced a plausible result.	1	2	3	4	5
Visual aids were easy to read and understand.	1	2	3	4	5
The team addressed authentic alternative scenarios and/or the need for future work.	1	2	3	4	5
I enjoyed the presentation; the presenter(s) held my attention for the full extent of the talk.	1	2	3	4	5
I would like to learn more about this team's solution method.	1	2	3	4	5

What is one question you would like to ask this team?

Additional questions or comments:

**TABLE D.5: A RUBRIC FOR PEER ASSESSMENT OF ORAL PRESENTATIONS OF MATHEMATICAL MODELING RESULTS.
(ADAPTED FROM GALLUZZO AND WENDT)¹**



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APPENDIX D

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The shared hope and vision of the Consortium for Mathematics and its Applications (COMAP) and the Society for Industrial and Applied Mathematics (SIAM) is that this report motivates the educational community to make a place for mathematical modeling in curriculum, from pre-K through undergraduate levels. Further, that the value and importance of the skills required to execute math modeling effectively - logical thinking, problem solving, sensitivity analysis, and communication to name a few - is recognized and nurtured. COMAP and SIAM, and the writing team, who donated their time and expertise to this report, and without which there would be no GAIMME, are proud to publish this first edition of the Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME) report.



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Heather Gould	Resources appendix
Ben Galluzzo	Assessment appendix
Sol Garfunkel	Preface

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Respectfully,
James Crowley, Executive Director, SIAM
Sol Garfunkel, Executive Director, COMAP
Michelle Montgomery, Director of Marketing and Outreach, SIAM

A partnership between SIAM and COMAP, Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME) enables the modeling process to be understood as part of STEM studies and research, and taught as a basic tool for problem solving and logical thinking. GAIMME helps define core competencies to include in student experiences, and provides direction to enhance math modeling education at all levels. A mix of professionals wrote and reviewed the sections to present various levels and perspectives. The GAIMME report is a freely downloadable report from both SIAM and COMAP's websites.

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Early and Middle Grades (K–8)
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AND INCLUDES

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Discussion of teacher implementation
Suggestions for assessment

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